The Mathematics Teacher

DEVOTED TO THE INTERESTS OF MATHEMATICS IN SECONDARY AND JUNIOR HIGH SCHOOLS

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THE MATHEMATICS TEACHER

VOLUME XVI FEBRUARY, 1923 NUMBER 2

OBJECTIVES IN TEACHING OF MATHEMATICS IN SECONDARY SCHOOLS[1]

By GERTRUDE E. ALLEN
Oakland, Cal.

We are living in a critical and intensely interesting period of educational development—a time of revaluation, reorganization, and reconstruction. It is our privilege to contribute to this work of reconstruction in the University High School and share in the pleasure and responsibility of planning new buildings and of shaping the aims and policies of the curriculum. In the clear light of our accepted objectives of education in a democracy, we have set at our work anew, and "each in his separate star" has taken a splash at the same ten-league cancas, to draw the thing as he sees it with whatever skill he may command—using the brushes of camel's hair or, if he is not so favorably endowed, the crude pencil or palette knife.

In regard to the present period of reconstruction one should feel optimistic. Procedure in educational circles has become more scientific, and there is encouraging evidence that "the transfer of improved efficiency" is a reality and that "ideals of the scientific method" are controlling the activities of educators. Evidences of an organized plan of action are clearly visible in the requiring of an agreement as to the definition of general aims in education and of specific objectives in different subjects, in the development of a teaching technique consistent with these objectives, and in the building of a curriculum in a clear field, restricted only by the contract of abiding by the accepted cardinal principles in education. The curriculum maker has agreed upon the following criteria: "Each subject and each item in the subject must justify itself—or negatively, no subject or item may be retained in any curriculum unless its value viewed in relation to other topics and

[1] Reprinted from the University of California High School Quarterly, October 1921.

to time involved can be made reasonably probable." In accordance with this thinking, the aim of the following discussion is to present the objectives in the subject of mathematics and to justify the place of mathematics in the curriculum of secondary schools.

Neither the redefinition of objectives in mathematics nor the proper placing of the subject in the curriculum can be satisfactorily made without a preliminary survey of the present status of mathematics and of certain influences which have brought about that status. The first observation in this survey is that most adults have strong personal prejudices either against or in favor of mathematics. Their convictions are usually based either upon their own failure or success in dealing with situations which require mathematical ability and training or upon early classroom experiences. Their ideals may have been formed under the inspiration of the rare teacher who "allured to brighter worlds and led the way," or, on the other hand, unfortunately they may have been determined by "The witless shepherd who persists to drive a flock that thirsts not to a pool disliked." Now the witless shepherd did enough evil in his own generation, yet his influence is perpetuated in the present attitude of children. How many times has a class been interrupted at the climax of interest in a problem by such questions as, "Why waste time on arithmetic when calculating machines can do the work?" "Why should a girl study mathematics if she never expects to earn her own living?" or "What use has a lawyer for geometry?" Young people never invented these ideas themselves, they are mere reechoes of opinions heard outside of school. It is highly desirable that parents and guardians leave children free to form their own prejudices and to evaluate the subject on its own merits.

The second observation is the melancholy fact that the percentage of failures in mathematics is higher than for any other subject. A high percentage of failure in any subject is a condition to cause deep concern whether that failure is due to difficulties inherent in the subject, to poor teaching, or to lack of effort on the part of the student. "Failure" is a word that should never enter the "bright lexicon of youth." One motive for the redirection of teaching in mathematics is to make wise adjustments, to lesson the number of preventable failures.

The third observation is that the prevailing courses of study in mathematics have developed under university domination. The first college entrance examination was set by Plato when he placed over his door the inscription "Let no one ignorant of geometry enter here." The organizers of secondary schools have been controlled too largely by a desire to satisfy university entrance requirements, with the resulting corollary that they imitated university methods and university text books, adapted only to the adult and ill-suited to the activities of the adolescent youth. Teachers have allowed the work to become too formal, too theoretical, too far removed from every-day use. Courses based on alleged needs in future courses have led to abstract content and manipulations of no value either at the immediate time or later. Traditionally, the minimum requirements in mathematics for university matriculation have been algebra and geometry; consequently, a year of formal algebra and a year of demonstrative geometry have been required for high school graduation. Recently the California State Board of Education and the University of California have arranged a readjustment of this plan, for experience has shown that the ruling was ill-advised both from the standpoint of the greatest good to the vast majority of students and from the best interests of the subject values themselves.

The last observation of this survey is that school authorities are satisfied to be guided by a blind belief in the theories of certain plausible educational propagandists, whose theories have nevertheless failed to stand the test of time. The chief offender against mathematics is the formalist with his dear departed doctrine of formal discipline. As a result of his wrong ideals and impossible objectives in education, mathematics has received so much unpleasant notoriety that it really has just cause for a damage suit. Indeed it is only slowly recovering from the injurious effect; it is not yet able to enter the race for popularity in the social order in which we live. While the belief in formal discipline prevailed, the writers of school text books in mathematics organized a very intricate philosophical system of theories and difficult problems. In arithmetic there was undue emphasis on involved computations, puzzles, and complex analyses; in algebra on the manipulation of symbols and the explanation of theo-

ries; in geometry, on finished logical forms for which immature minds were unprepared. In the study of these text books, certain exceptional minds found delight and were challenged to successful effort, but mathematics was popularly regarded as a stern disciplinarian, occupying a throne of high authority in school life only, but of slight importance in the life of the world. The effect of dethroning the doctrine of formal discipline should have been to displace also its strongest ally, mathematics. If the psychologist and the pedagogue had possessed the courage of their convictions in regard to disciplinary values, why should they not have operated under the spoils system and have found time during a whole generation to throw out the entire troublesome business of mathematics from the curriculum? Truth forces the answer upon us that mis-shape them as we will, number relations of time, space, and quantity constitute one of the fundamental concerns of human life. The power to decide wisely the demands, "where," "when," "how many," "how much," has always been essential to man's progress, and as civilization grows more complex, an active interest in the quantitative side of life and the power of intelligent control over the same become even more necessary. When vital needs and activities, essential powers, and noble ideals are made the basis of teaching aims, of text book content, and of courses of study, mathematics will fill its rightful place in the scheme of modern education. The skeleton of formal discipline may then be forever locked in its closet and forgotten.

The utilitarian group is another set of propagandists dangerous to the welfare of mathematics. Although the formalist is dead and belongs to ancient history, yet there are many modern extremists who are alive and flourishing. The empiricist and the "projectionist" belong to this order. Their popular doctrines cannot be accepted, if educators are truly consistent, until weighed in the scientific balance and evaluated. The empiricist, in his non-analytic mind, has a notion that "men go up in flying machines made of wood, wire, complex fractions, gasoline and exponents." What he wants is the formula for mixing the ingredients that he may obtain the finished products, but he cares no more for the knowledge that produced the formula than does the machine he operates. These mechanical results of mathe-

matical skill are very important in the machinery of civilization and satisfy the demand of the four million, but it should not be the only objective in education or the race will enter upon its decadence. Guardians of progress must provide an opportunity for the four hundred to be educated for intellectual leadership. They must also furnish a curriculum that will enable all pupils to discover their dominant interests that they may eventually with more certainty find their place in society. In a recent number of the Atlantic Monthly, Dr. G. M. Stratton very aptly expresses this need of broad general training.

> If one is to think effectively of sugar beets or airplane engines, he must study such beets and such engines. But he will not think effectively upon these if he thinks of these alone: His interest and his knowledge must widen to the principles of agriculture or of aerodynamics; and beyond, he will need botany or physics and chemistry. Chemistry then is important for a lad uncertain whether he will deal with beets or engines. But what of the boy or girl who does not know and whom no one as yet can tell whether beets, engines, taxation, tuberculosis, or the Gospel will lie at the center of his thinking in the time to come? He had best be at home in the central studies into which all special subjects lead; and with these and even more useful, he will need habits of intellectual economy, of accuracy in interpreting what is read and heard, of distinguishing important from unimportant, of throwing himself with vigor into the work in hand—these are a part of intellectual training.

The "projectionist" finds true education only in "whole-hearted, purposeful activity in a social setting." But the theory of this "project method" as explained by Dewey or Kilpatrick fails to find adequate interpretation in many so-called projects operated by hobby-riding enthusiasts. Little educational value can be found in a multiplicity of physical activities alone, for mental development may be arrested on the plane of concrete thinking. Spectacular material results too often blind the operators to the most vital part of the real project—rational analysis, mastery of fundamental principles, the formation of ideals, and the development of power to meet new situations. Closely allied with the idea of whole-hearted activity is the demand for constant excitement and entertainment prevalent among our young people of today. Any form of educational work that involves sustained mental effort becomes unpopular unless it is so skillfully camouflaged that it can be made to tempt even the non-intellectual and the lazy. We are in danger of developing a soft pedagogy that brings a maximum of temporary pleasure and a minimum of intellectual attainment. Artificial standards in

school life may bring a good time then, but are sure to bring a bad time later. The fruit of the ideal "play way" is joy in conquest, pride in mastery of the difficult task, and rejoicing in the beauty of daily routine work even as a "strong man to run a race." Education should harness the play spirit so that all of its energy is not spent in the irresponsible frolics of youth, but a due portion of it diverted toward the disciplined labor of manhood. Under the control of an inspiring teacher young people can be made to take the same keen interest in learning to work efficiently that they show in playing whole-heartedly.

The purpose of this survey has been to present significant facts relating to the present status of mathematics in the public schools and the trend of present changes. Let us return now to our main purpose, the justifying of the place of mathematics in the high school course. A definition of aims should precede any course of study in mathematics since one's aims must determine the selection and use of material; in like manner the central aims of the whole system of education determine the criteria by which we must justify the place of mathematics. The teachers of the University High School have adopted the aims stated by the Commission on Reorganization in their report on Cardinal Principles in Secondary Education:

> Education in a democracy, both within and without the schools, should develop in each individual the knowledge, interests, ideals, habits, and powers whereby he will find his place and use that place to shape both himself and society toward ever nobler ends. This commission, therefore regards the following as the main objectives of education: (1) Health, (2) Command of fundamental processes, (3) Worthy home-membership, (4) Vocation, (5) Citizenship, (6) Worthy use of leisure, (7) Ethical character.

The mathematics department has adopted the aims formulated by the National Committee on Mathematical Requirements as follows:

> The primary purposes of the teaching of mathematics should be to develop those powers of understanding and analyzing relations of quantity and of space which are necessary to a better appreciation of the progress of civilization and a better understanding of life and of the universe about us, and to develop those habits of thinking which will make these powers effective in the life of the individual.

This statement of aims in mathematics is in perfect accord with the central aim for the entire system of education. We shall endeavor to interpret it in terms of the specific branches of mathematics and to make it the basis of a reorganized course of study for both junior and senior high schools.

Any individual's direct use of mathematics depends upon his occupation. Every member of the community, not a dependent, has a limited bread and butter need for its use; as for example, he must be able to count change, to foot up a bill, to tell time, and to use the common weights and measures. Beyond this use, indispensable for self protection, the high school aims to develop a better understanding of the forces in nature and a better appreciation of the inventions and improvements of civilized man. With reference to the extent of their need for mathematics, people whom the high school can help may be considered in four groups: (1) the average intelligent citizen, who, apart from any vocational need, should understand modern business practice, should be informed about public utilities and the expenditure of public money, about the world's commerce and industries, and should have some intelligent idea of how the surveyor calculates distances, the astronomer explores the heavens, the navigator finds his position at sea, or the aviator determines his elevation; (2) the mechanic or tradesman, whose work requires a limited but specific knowledge; such, for example, as the surveyor's use of his tables, the mechanic's use of formulas, the sheet metal worker's use of geometric construction; (3) the engineer, whose technical work requires extended knowledge of practical mathematics; (4) the specialist, who studies pure mathematics for use in scientific research, with an idea of teaching it or merely for the intellectual joy of doing it.

Mathematics teachers of the University High School agree that the main objectives will be achieved most directly by uniting these four groups in the junior high school and differentiating them in the senior high school, recognizing in so doing the principle that the work of the former period is essentially child-centered, and of the latter, more definitely subject-centered. For the junior high school we recommend an extensive three-year course in General Mathematics, of such content and thoroughness as to be of direct value to the average individual in later life and within the range of the pupil's interest and understanding. We further recommend that, as a rule, all students should be expected to continue this course through the ninth school year, but should not be subject to uniform requirements in mathematics thereafter. For the senior high school we recommend

that intensive courses in algebra, trigonometry, demonstrative and analytic geometry be provided, and that these courses be logically organized and rigorous, designed for the mathematically inclined and the mathematically capable.

THE JUNIOR HIGH SCHOOL

The specific aims in the junior high school are, first, to give instruction and training in the mathematics useful to the average intelligent citizen; second, to disclose mathematical ability or the lack of it, so that pupils may be guided in their choice of later work. In order to fulfill these aims, the teacher must have exceptional knowledge of his subject and of its applications, and be able to adapt his work to the changing interests and maturity of the pupils.

The traditional division of mathematics into subjects which resulted in teaching arithmetic only in the seventh and eighth grades, formal algebra in the ninth, and geometry in the tenth should be discontinued. In place of this the three years' work should be regarded as a unit, and should concern itself only with the most practical of problems, selected because they occur in social activities common to the lives of most men and women. In connection with these problems, thorough training should be given in the most economical processes for solving them; such as, short cuts in arithmetic, the equation, the formula, the graph, and the standard reference tables. About one semester of time can be gained by omitting obsolete topics and unusual complications in arithmetic and the more formal and involved parts of ninth year algebra. The time gained may be well used for work in geometric drawing and form study in connection with mensuration, and for very elementary problems in trigonometry. The work in geometry and trigonometry should be of the simplest form, and need take only enough time to give some appreciation of and insight into the nature of the subject and its many uses. Experiment has shown that this tangible, practically applied work arouses more enthusiasm and interest than any other part of the junior high school mathematics.

Guidance for later work should be an important responsibility for the junior high school teacher, to the end not only that civilization may profit by all available mathematical talent, but also

that the non-mathematical may be saved from failure to direct their energies toward work for which they do have taste and ability. The teacher should make himself a reliable source of information concerning mathematical requirements of the different trades and professions; he should be familiar with the State Board of Education and University regulations affecting mathematics, and should advise students accordingly. At the end of the junior high school period, the pupils should be prepared and eager to go on to higher schooling or ready to go out into life better informed as to what is going on around them and more keenly alert to the work of the world.

THE SENIOR HIGH SCHOOL

The specific purpose in the senior high school is to train those who will have some definite need for technical mathematics. The standard in these classes should be very high; furthermore, the quality and extent of the work should be such that the students will be adequately prepared for more advanced work in pure mathematics, science, or engineering at a university. The prerequisites for higher work are algebra and geometry broadly considered; that is, in a sense broad enough to include solid geometry, trigonometry, and analytic geometry. Algebra has the wider logical development and organization. Trigonometry and analytic geometry, while dealing with subject matter essentially geometric, combine the methods of algebra and geometry in the solution of problems. All these subjects lose much of their value by artificial isolation from each other or from other activities, and gain by considering mathematics as an integrated whole with one fundamental aim—to analyze and interpret relations of quantity and space.

The function of algebra in mathematics is well expressed by Nunn: "The aim of algebra, including trigonometry and analytic geometry, is to develop a calculus, that is, a system of symbols and rules for the manipulation of those symbols, by means of which the investigation of some definite province of thought or of external experience may be facilitated." Innumerable illustrations can be found of the use of the highly specialized tools of algebra which either make possible or very greatly simplify the solution of problems of business, engineering, and

scientific research. The electrician, the aviator, the navigator, the mechanic, the statistician, the insurance actuary, and the business man can do his work more efficiently on account of the vast array of formulas developed by algebraic method and interpreted for practical service. Problems like the following deserve more attention than we give them: If a home buyer assumes an indebtedness of five thousand dollars, what monthly payment including interest will enable him to clear up the debt in five years? Or, how long will it take the same man to clear up his debt if he can pay seventy-five dollars a month? (2) If a man opens a savings account at the age of twenty and adds to it regularly fifty cents for every working day, will it, as a prominent savings bank recently advertised, amount to fifteen thousand dollars when the man is sixty years old? It is extremely tedious or quite impossible to solve these problems by arithmetic, but a short and simple process to solve them by the correct algebraic formula or by the use of amortization and annuity tables. A great auditorium is an architectural triumph or a monumental failure, and a sad waste of public money according to whether or not the architect understands the equation and the construction of the curves employed, and can place the stage at the geometric focus. The bad acoustic properties of many a great public hall are silent proof of the architect's insufficient technical training. The remarkable barrage maneuvers of the world war were made possible by a practical application of the equation of the parabola to the path of projectiles. It is not necessary to multiply illustrations. The important thing to keep clear in this work is that it is essential to develop skillful technique in the algebraic processes, but that algebra is in no sense a game of symbol juggling. The symbols in every case refer to realities beyond themselves found in the material world, and the value derived from working with these symbols lies in the practical application of the result to problems of human welfare.

The specific function of the study of geometry is, first, to make the student familiar with the fundamental facts regarding form and space; namely, the basal propositions of plane and solid geometry, second, to develop an understanding and appreciation of a deductive proof and the ability to use the method of logical

reasoning in any situation where it is applicable; third, to form ideals of the scientific method and of methods of procedure not identified with the scientific method, but which may result from specific discipline.

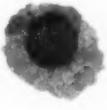
Although the time spent in studying algebra is abundantly justified by its direct practical values to selected groups, we can make no such justification for spending a year on demonstrative geometry. We recognize the practical importance to everybody of knowing the fundamental geometric facts, but common sense observation and repeated experiment in the every day affairs of life bring a conviction and understanding of these facts quite as satisfying to the majority of minds as a logical proof. The students in the junior high school become familiar with the geometric facts of general application, and even the more technical theorems and formulas essential to the engineer can be mastered by rote in a few weeks. We believe in the educational possibilities of the study of geometry but do not pretend to defend a place for it in the high school curriculum on the basis of direct utilitarian values only. At the risk of becoming a welcome target for the utilitarian educator we shall make the claim for demonstrative geometry primarily as an exercise in logic, as a means of mental training, and as a medium for developing high ideals of accuracy and truth. In opposition to those who would train for specific utility only, we have sufficient authority safely to maintain that there is far greater educational value in the power to think through a new problem for one's self than in acquiring rote knowledge of time-honored facts in the mastery of important general principles than in the perfection of certain specific automatic skills. The subject matter of geometry exemplifies most typically and clearly certain methods of procedure of the greatest value in cultivating such power. We understand that it is useless to expect "volunteer growth" of this power from mere lesson memorizing and reciting; that on the other hand the teacher must aim to cultivate it with scientific care. He must make the student definitely conscious of method; abstract the process from the subject of geometry and furnish the opportunity for much practice in applying the principles of organization and reasoning to non-mathematical problems of life. When we teachers of geometry learn to handle the

subject with sufficient wisdom and skill to realize its potential educational values, it will no longer be necessary to apologize for our existence of continually to have to justify the presence of the subject of geometry in the high school curriculum.

In the treatment of aims in geometry, there is another point which we do not wish to evade—the case of students who take it on account of arbitrary college or graduation requirements. It is a mistake to prescribe geometry for all students; for those not gifted with the power of abstract reasoning there is much greater educational value in other activities. About girls and geometry there is much foolish sentiment. Girls as a whole play the game quite as well as boys, and further more it is hardly in keeping with the modern ideals of the emancipated woman that girls should deliberately acknowledge for men an exclusive intellectual aristocracy. However, as enthusiastic teachers, we deplore the idea that either boys or girls should be driven into the geometry class like lambs to the slaughter. On the other hand, it is expedient to exercise arbitrary authority over those students who are definitely preparing for a university, and, further than that, strongly to advise the geometry course for such other students as offer promise of developing later into university material. No matter whether the vocational adviser believes in mental training or not, he owes it to the student to send him out from the high school properly equipped to meet the conventional requirements which he will encounter.

In addition to these last mentioned courses of the conventional type, we recommend also for the senior high school a course in socialized arithmetic. This arithmetic should be of very practical value and of interest to both boys and girls, whether they expect to go to a university or not. It should deal with personal and family business affairs, such as simple household accounts and budgets, business laws and conventional business forms, and some of the larger aspects of social, industrial, and economic life. Thrift as a patriotic necessity should be urged throughout this work. The principles and problems should emphasize thrift in time, money, and natural resources. This course in arithmetic should be elected late in the high school work in order that the student may regain the skill in figuring he had acquired at the age of twelve and retain it until he needs it in life beyond the

school, also that he may have sufficient maturity to appreciate the meaning of and necessity for "worthy home membership" and "intelligent citizenship." Practical work of the kind outlined in the following course would, we think, not only bear such valuable fruits as a more rational basis for financing the home, greater efficiency in private business, more thrift in providing against dependence in old age, and a responsible attitude toward public duties, but it would also divert the attention of critics from the fact that there is apparently no immediate cash value in the market place for cultural and disciplinary outcomes. Such a course as this will, we think, help teachers of mathematics to work more whole-heartedly at their project of developing the greatest possible practical and disciplinary values of their subject; it will also help them to realize with Major Huntington, one of the leaders in the educational work of the world war, that "there is a great future for mathematics in this country, especially accurate arithmetic."



OUR GEOMETRY IN EGYPT AND CHINA

By WILLIAM A. AUSTIN
Fresno High School, Fresno, Cal.

Two articles describing the laboratory plan of teaching plane geometry in Fresno High School have appeared in School Science and Mathematics. The first article, "A Flu Dream in Mathematics," was in the November, 1919 issue. The second article, "A Dream Come True", was published in the October 1921 number.

Letters from hundreds of teachers all over the country, asking for additional and detailed information about this plan of teaching the subject and about the Manual of Plane Geometry, have been inspired by these articles. In reply to each inquiry samples of work done by our students, pages of the Manual, and other material giving an idea of the scheme have been sent.

In addition to these, several letters have been received from distant lands. One came from the professor of mathematics in Assiut College, Assiut, Egypt; one come from a teacher of mathematics in the high school at Bombay, India; one came from the instructor of mathematics in the *Ateneo de Manila,* Manila, P. I.; one from the master of mathematics in a State High School in Tasmania, Australia; and one from the professor of mathematics in Yao Jen Academy, Ning-po, China.

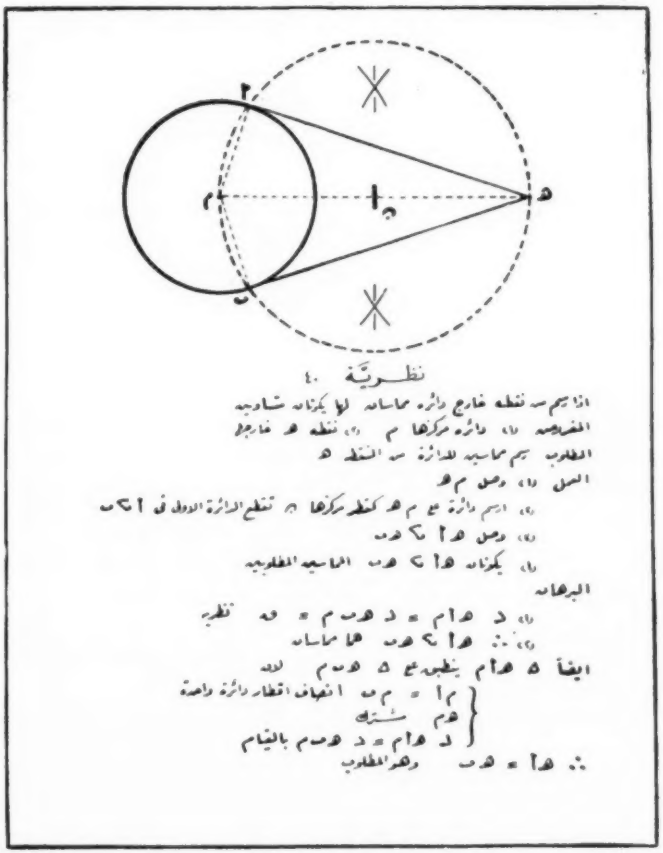
A complete set of plates made by our students and a mimeograph copy of the Manual were sent to these distant teachers. They were invited to try the scheme, to write us concerning the results of their efforts, and to send us work done by their students. Replies from these distant friends have given the incentive for this article. The drawings accompanying this discussion were made by students in two of these far-away schools.

Figure 1 is one of our regular plates. The drawing from which this is reproduced was made by a student in Assiut College, Assiut, Egypt. The lettering and the demonstration written on this plate are in the Arabic language, the translation of which is as follows:

Two tangents drawn to a circle from an external point are equal in length.

Given: (1) Circle O.
(2) Point P outside the circle.

To construct two tangents thru point P to circle O.



Theorem 40

Construction:

(1) Draw PO.
(2) Construct M, the mid-point of PO.
(3) Draw a circle with M as the center and with MO as the radius, intersecting the given circle at A and at B.
(4) Draw PA and PB.
(5) PA and PB are the required tangents.

To prove PA and PB tangents to circle O.

Proof: (1) $\angle PBO = \angle PAO = 90°$ . . . . . Th. 39 Cor. 1.

(2) Hence, PA and PB are tangents. Th. 36.

To prove $PA = PB$.

Proof: (1) $\triangle PBO = \triangle PAO$, by Th. 2 Cor. 1, since,

(a) Both are right triangles.

(b) $PO = PO$, and

(c) $BO = AO$, radii of the same circle.

(2) Hence, $PA = PB$. . . . . Par. 73.

The directions given in the Manual for the construction of Plate 32 from which the drawing in Figure 1 was made follow:

PLATE 32

Tangents to a Circle from an External Point

Construction: Fig 1. Draw circle O. Select point P outside the circle. Draw PO. Construct M, the mid-point of PO. Draw a circle with M as a center and with MO as a radius, intersecting the given circle at A and at B. Draw PA and PB.

Is PA perpendicular to AO? Why? Is PB perpendicular to BO? Why? Are PA and PB tangents to the circle? Why?

Measure PA and PB. Are they equal? Were they made equal?

Theorem XL. *Two tangents drawn to a circle from an external point are equal in length.*

Three things are to be done: (1) To construct two lines thru point P tangent to circle O; (2) To prove that these lines are tangent to the circle; and (3) To prove that these tangents are equal in length.

Demonstration: (Note: The formal demonstration given in the Manual is about the same as that written on Fig. 1.)

Figure 2, which is one of the six problems of construction of our Plate III, was made by a student in Yao Jen Academy, Ning-po, China. The discussion, which is written in the Chinese language, is translated as follows:

Given: Line AB.

To bisect AB and to erect the perpendicular bisector of AB.

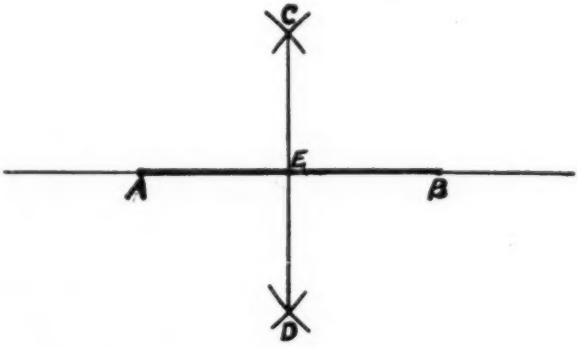
Construction:

(1) Around A and B as centers with a radius greater than one-half AB strike two pairs of arcs intersecting at C and D.

(2) Draw CD, cutting AB at P.

(3) P is the mid-point of AB. CD is the perpendicular bisector of AB.

Problem 1. Yeh Li Hua.



1. 已知: AB直線.
2. 求作: (1) AB之中点.
(2) AB之垂直二等分線.
3. 作法: (1) 以B爲圓心, 大于AB之半爲半徑, 各規一弧, 相交于C, D兩点.
(2) 連結CD, 交AB于E.
(3) E即AB之中点, 而CD即AB之垂直二等分線也.

The directions given in the Manual for the construction of Plate III Fig. 1, from which Figure 2 was made are as follows:

PLATE III

Problems of Construction.

Six Problems of Construction are as follows: (1) To bisect a given line and to erect the perpendicular bisector of the line; (2) To erect a perpendicular to a given line at a given point in the line; (3) To construct a perpendicular to a given line from

a given point not on the line; (4) To bisect a given angle; (5) To lay off a given angle in a given position; and (6) To construct thru a given point a line parallel to a given line.

To Bisect a Line and to Erect the Perpendicular Bisector of the line. To bisect a line means to find by construction the midpoint of the line.

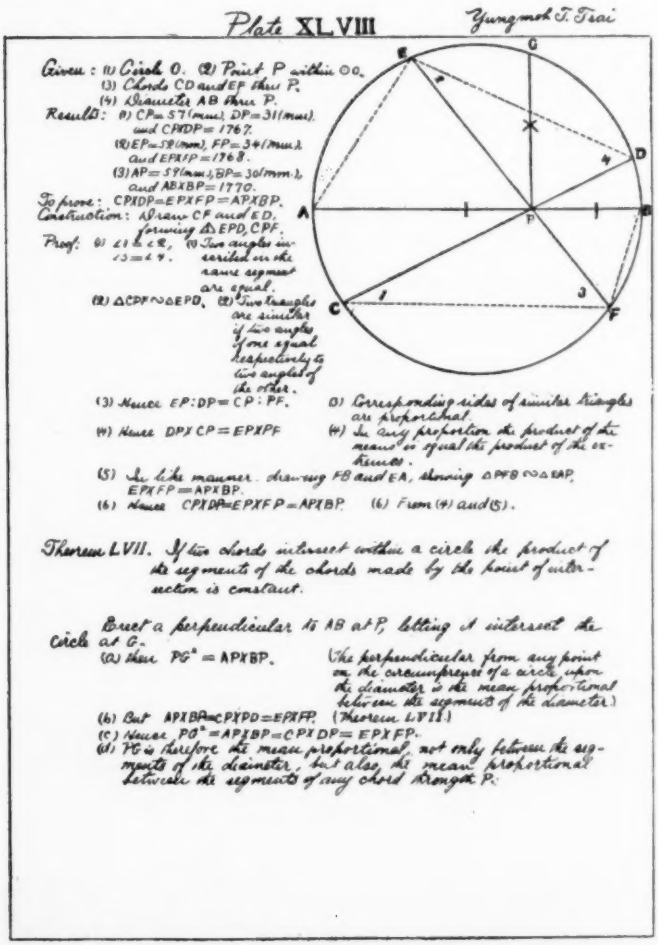


Plate XLVIII
Yungmoh T. Tsai
Given: (1) Circle O. (2) Point P within ⊙O.
(3) Chords CD and EF thru P.
(4) Diameter AB thru P.
Result: (1) CP = 57 (mm.), DP = 31 (mm.), and CP×DP = 1767.
(2) EP = 58 (mm.), FP = 34 (mm.), and EP×FP = 1768.
(3) AP = 59 (mm.), BP = 30 (mm.), and AP×BP = 1770.
To prove: CP×DP = EP×FP = AP×BP.
Construction: Draw CF and ED, forming △EPD, CPF.
Proof: (1) ∠1 ≅ ∠2, ∠3 = ∠4.
(1) Two angles inscribed on the same segment are equal.
(2) △CPF ∾ △EPD.
(2) Two triangles are similar if two angles of one equal respectively to two angles of the other.
(3) Hence EP : DP = CP : PF.
(3) Corresponding sides of similar triangles are proportional.
(4) Hence DP × CP = EP × PF.
(4) In any proportion the product of the means is equal the product of the extremes.
(5) In like manner, drawing FB and EA, showing △PFB ∾ △AP, EP × FP = AP × BP.
(6) Hence CP×DP = EP×FP = AP×BP.
(6) From (4) and (5).
Theorem LVII. If two chords intersect within a circle the product of the segments of the chords made by the point of intersection is constant.
Erect a perpendicular to AB at P, letting it intersect the circle at G.
(1) Then PG² = AP×BP.
(The perpendicular from any point on the circumference of a circle upon the diameter is the mean proportional between the segments of the diameter.)
(2) But AP×BP = CP×DP = EP×FP. (Theorem LVII.)
(3) Hence PG² = AP×BP = CP×DP = EP×FP.
(4) PG is therefore the mean proportional, not only between the segments of the diameter, but also the mean proportional between the segments of any chord through P.

PLATE XLVIII

Construction: Fig. 1. Draw line *AB* representing a distance of 250 miles. With *A* as a center and with a radius greater than one-half *AB* strike arcs above and below *AB*. With *B* as a center and with the same radius strike arcs intersecting the other arcs at *C* and at *D*. Draw *CD*. Mark the point of intersection of *CD* with *AB*, *P*. Point *P* is the mid-point of *AB*. Line *CD* is the perpendicular bisector of *AB*.

Figure 3 is our Plate XLVIII. It was made by a student in Yao Jen Academy. A copy of the directions for the construction of this plate follows:

Segments of Intersecting Cords.

Construction: Fig. 1. Draw circle *O*. Select point *P* within the circle. Draw thru point *P* two chords *CD* and *EF* Draw also thru point *P* diameter *AB*.

Measure *CP*, *DP*, *EP*, *FP*, *AP*, and *BP*. Compare the following products: (1) $CP \times DP$; (2) $EP \times FP$; and (3) $AP \times BP$. Are these products equal?

In these computations the products of numbers two at a time equal the same result. When such is the case, the products are said to be *constant*. It appears that the product of the segments of chords intersecting within a circle is constant.

Theorem LVII. If chords intersect within a circle the product of the segments of the chords made by the point of intersection is constant.

Demonstration: (Note: The demonstration given in the Manual is nearly the same as that usually given in any text. The additional item is headed *Results*. Under this heading the student places the numerical results of his measurements and computations).

Additional Study of Plate XLVIII. Erect a perpendicular to diameter *AB* at point *P*, letting it intersect the circumference of the circle at *G*. Show that *PG* is the mean proportional, not only between the segments of the diameter, but that it is also the mean proportional between the segments of any chord drawn thru point *P*.

Exercises on Plate XLVIII.

1. If $FP = 24$, $EP = 36$, and $DP = 16$, find CP.
2. If $BP = 12$, $CP = 21$, and $DP = 16$, find AB.
3. If $AB = 42$, $FP = 15$, and $EP = 24$, find AP.
4. If $BP = 7$, $AP = 16$, and $CD = 22$, find CP and DP.
5. If $FP = 12$, $EF = 30$, and $AB = 35$, find how far point P is from the center of the circle.
6. The radius of a circle is 25. The shorter segment of a chord thru point P 17 inches from the center is 14 inches. Find the length of the chord.
7. If $PG = 24$ inches, CP is four times DP, and DP is 4 inches longer than BP, find CD and the radius of the circle.

The laboratory plan of teaching plane geometry has been in operation in Fresno High School three years. The Manual, which was prepared for class room use, contains complete directions for the construction of all drawings, for the study of some specific fact which each construction is made to illustrate, for the formal demonstration of apparent truth, and for the formation of definitions.

This work includes the preparation of seventy-two regular plates, many exercise plates, and the solution of numerous problems. All the essential theorems and corollaries given in any text on the subject, and especially those recommended by the National Committee on Mathematics, are in the Manual.

On our program plane geometry is scheduled for seven periods each week. Double periods twice each week are usually used for the construction of plates. During these laboratory periods the teacher is engaged in assisting and instructing students individually or in groups or in checking and correcting work handed in. These periods are very informal. Students are given privileges usually allowed in a laboratory. Assisting and consulting each other in their work is not only permitted but encouraged. Frequently a student who is in trouble is directed by the teacher to inquire of another student whom she knows is able to help. Leaders are discovered by the pupils; and, just as in any community, no teacher is needed to point them out.

With the Manual before him the pupil is able to execute the constructions with his drawing tools. The directions are so

clearly stated and so specific that he may complete a drawing, discover the truth embodied in the construction, and comprehend the argument in the demonstration with little or no aid from his teacher.

Plate XLVIII, which is given in Figure 3 above and the copy of the directions for its preparation, tipify the method of procedure with nearly all theorems. Students are required to take the following steps in the study of a proposition: (1) a construction is made according to directions; (2) measurements are taken; (3) numerical computations are performed; (4) attention is directed to an obvious truth; (5) this truth is stated in the form of a theorem; (6) the formal demonstration of the theorem follows; and (7) the exercises involving the truth of the proposition are solved.

All definitions are developed by constructions before a formal statement is attempted. All fundamental concepts, axioms, and postulates are introduced in a similar manner.

Recitations are conducted during the other periods each week. There is no uniform plan of procedure for these recitations. By observing in many classes a visitor may see the entire class following the discussion of one student, a small group in recitation or receiving instruction, or even a single student reciting before the teacher. In all these recitations one distinctive characteristic is noted: students tell what they have done and what they know. One never sees a pupil reading material written on the board from memory and pointing haphazardly at a drawing.

We have not altered the subject-matter; we have merely reversed the method of attack. Experience has taught us that mental activity of students of this age in high school can be stimulated into action easier thru the avenue of muscular activity than by way of eye, ear, or reason; that they have an abundant supply of muscular energy demanding use and a minimum supply of thought energy difficult to reach and to bring into action; and that they are by nature scientists living and experimenting in a world-laboratory twenty-four hours each day. We also believe that geometry is a laboratory science, the natural approach to the study of which is scientific. We,

therefore, conclude that the science of plane geometry should be studied in a scientific manner by students who are themselves scientists.

One advantage of this plan over the usual method followed in plane geometry is the fact that students are always actively at work and that they take the initiative. There is something to do every moment and they know just what it is. No teacher is required to direct them nor to tell them when to begin. They are interested and happy because they are exercising their natural instinct, a supre-supply of energy demanding expression in activity. They are doing things.

Students may display individual ability and speed. There is no lock-step action; no class delay due to some dull pupil; and no helpless groping in darkness to keep the pace set by teacher or class. A class in geometry is a small community at work; the hum of industry—not noise—is heard.

The greatest argument in favor of the laboratory plan, however, is found in the method of attack. The student realizes the natural unfoldment of geometric truth in its embodiment. He sees the construction in its logical development. The steps he takes in the study of a theorem, if generalized, correspond almost identically to the steps taken by the human mind in its investigation of new truth and in its satisfaction of the validity of any fact discovered.

Permit me to quote a paragraph from professor Tsung-woo Tsha's letter. He is my friend in Ning-po, China. He writes:

"I am deeply impressed with the admirable work of your students. The plates amply bear out the truth of your statement that the subject-matter taught and the method of attack conform to modern ideas of education and life. This principle is seldom observed in ordinary text-books. The common method of instruction places the student in a passive attitude; and, consequently, little can be obtained from the lectures of the teacher. Under your system, however, the student is led gradually to depend not only on his eyes and ears but on muscular exertion so as to develop spontaneously the basic concepts and principles of geometry. This naturally creates an interest which effectually takes the 'dryness' out of the subject."

NOTE: The writer will welcome any letter containing criticisms, suggestions, or questions.

MECHANICS

By GORDON R. MIRICK
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Introduction

The subject of mechanics and its application to engineering science opens up a possible source of problem material for high school mathematics. Such ideas as velocity, acceleration, angular motion, momentum, force and energy give rise to some problems that can be used in place of some that are already in our text-books. A great many problems can be gotten together on statics and simple machines that are very elementary and worth while.

The elementary notions of mechanics of solids that are applicable to high school mathematics can be gotten together in about eight papers, as follows:

1. Rectilinear Motion of A Point.
1. Rectilinear Motion of A Point. Gravity.
1. Angular Motion. Applications.
1. Newton's Laws and Momentum.
1. Force and its Application.
1. Energy and its Application.
1. Simple Machines.
1. Fundamental Principles of Statics and Their Application to The Simpler Problems of Engineering Science.

Divisions of Mathematics.

The idea of space is fundamental in geometry. In mechanics we add the idea of motion, time, and mass to that of space.

If we add the idea of motion to that of space we get what is known as the geometry of motion. But with this idea of motion we have the idea of time and with these three ideas we get that branch of mechanics, known as kinematics.

The introduction of mass leads to numerous new ideas such as momentum, force and energy. This branch of mechanics is called dynamics. We divide dynamics into two parts—statics and kinetics.

In statics the forces acting on a body produce no motion. It is the science of equilibrium.

Kinetics treats in the most general way the changes of motion produced by forces.

We find different names for mechanics when it is applied to different subjects, as for example: thermodynamics as applied to heat; electrostatics and magnetostatics as applied to electricity and magnetism; hydrodynamics as applied to the flow of water; celestial mechanics as applied to astronomy; etc.

Part I: Kinematics.

I. Rectilinear Motion of a Point. Uniform Motion.

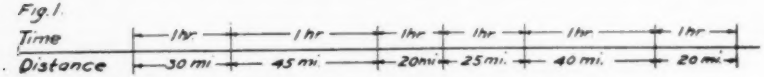
Suppose that an automobile left New York for Albany and arrived at Albany in 6 hours. They covered 180 miles by the way they went.

At Albany one of the party asked: "How fast did we go?" Some one answered, "30 miles an hour."

Now the question is: What did they mean by, "30 miles an hour?" They could not have gone at the same speed all the way for they must have had to slow down through towns and at railroad crossings. They must have divided the distance that they covered by the time taken to cover that distance to get 30. This 30 must represent the average speed in miles per hour.

Now let us suppose that they went 30 miles the first hour, 45 miles the second, 20 the third, 25 the fourth, 40 the fifth, and 20 the sixth. Their trip will look something like this:

Fig. 1.



$$\text{Average speed} = \frac{30 + 45 + 20 + 25 + 40 + 20}{6} = 30$$

$$= \frac{\text{Distance (total)}}{\text{Time (total)}}$$

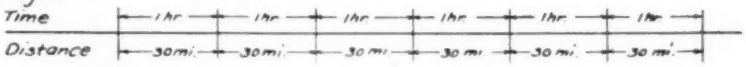
In the above, we have used the word speed. Now speed means the distance covered per unit of time without regard to its direction. But in an example like this we think of their going in a certain direction, so we should have used the term velocity. Velocity is the rate of motion in a given direction. It is a vector quantity.

If v_1 = velocity the first hour, v_2 = velocity the second hour, etc., we have,

$$\text{Average velocity} = \frac{v_1 + v_2 + v_3 + v_4 \ldots . v_t}{t}$$

Now let us suppose that they could have gone at the same speed, of 30 miles an hour, all the way. We will represent their trip by means of a new figure.

Fig 2.



We see from the figure that in every hour they covered the same distance of 30 miles. To get the total distance covered we multiply the distance covered per hour by the time.

Total distance = distance per hour $\times$ time.

or $s = v \cdot t$

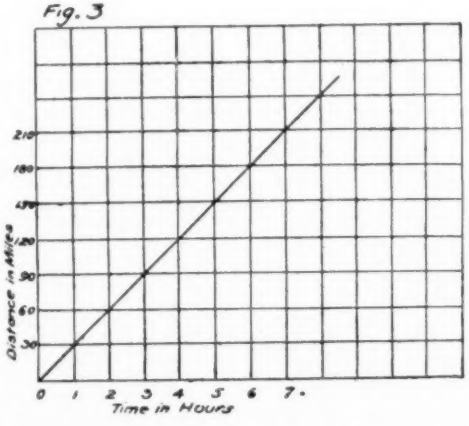
where, s = distance (total)

v = velocity (uniform)

t = time

We have here a case of uniform velocity, for the space covered per hour is the same or 30 miles.

Fig. 3



If we plot distance on the ordinate and time on the abscissa, we have a space curve for uniform velocity of 30 miles an hour. The graph is a straight line.

If they have been going 30 miles an hour, they must have been going a certain number of feet per minute and also a certain number of feet per second. Let us see if we can find these numbers.

30 miles $= 30 \times 5280$ feet; now if we divide 30×5280 by 60, we will get the number of feet per minute or 2640 feet per minute. If we divide 30×5280 by 60×60 or 3600, we will get the number of feet per second or 44 feet per second.

Let us sum up our ideas in the form of definitions.

Motion is the continuous process of change of position.

Rectilinear motion is motion in an unvarying direction.

Velocity is the time rate of change of position, or the distance passed over per unit of time in a given direction.

Uniform velocity is a constant time rate of change of position, or velocity such that the distance passed over in every unit of time is constant in magnitude and direction.

Problems.

Note: In these problems and all other problems on motion we consider only uniform velocity and uniform acceleration.

1. Compare the following by reducing all to feet per second (ft./sec.): (a) a man walking 4. M./h.; (b) a train going 45 M./h.; (c) a horse trotting a mile in 2 min. 10 sec.; (d) an automobile going 930 meters per minute.

2. What is the distance between two cities if it takes an automobile 4 hours to make the journey at an average speed of 20 miles an hour? Ans. 80 miles.

3. If a train is traveling at a constant speed of 55 miles an hour what distance does it cover in feet in 5 minutes?

Ans. 24,200 feet.

4. Compare the velocites of two bodies which move uniformly, one through 180 feet in one-quarter minute, and the other through 1¾ yards in three-eights of a second. Ans. 6:7.

II. Rectilinear Motion of a Point. Uniform Acceleration.

We have assumed that this automobile moves uniformly in any one hour, 30 miles an hour the first hour, 45 the second, 20 the third, etc. But what happens from the time it starts from rest until it has obtained a velocity of 30 miles an hour? Two

things could have happened; first, a gradual increase up to 30 miles an hour; second, increased slowly than fast than again slowly, etc. In both cases we have the velocity changing or we have accelerated motion. In the first case we have uniform acceleration and in the second case variable acceleration.

Suppose in this second case the velocity increased 6 feet the first second, 10 the second, 8 the third, 4 the fourth, 6 the fifth, 2 the sixth, 2 to seventh, 5 the eighth and 1 the ninth. This would increase the velocity to 44 feet per second or 30 miles an hour. The average or mean acceleration is $6 + 10 + 8 + 4 + 6 + 2 + 2 + 5 + 1$ divided by 9 which equals 4 8/9 feet per second per second.

$$\text{Mean acceleration} = \frac{\text{increase of velocity}}{\text{time taken for increase}}$$

Let us look at uniform accelerated motion which we are concerned with here.

This automobile is advertised to "accelerate" 6 feet per second per second. To accelerate 6 feet per second per second means that the velocity increases 6 feet in every second. Let us see how fast it would be going at the end of 7 seconds.

$v = 0$ velocity at the start

$v = 0 + 6 = 6$ velocity at the end of first second
$v = 0 + 2 \cdot 6 = 12$ velocity at the end of second second
$v = 0 + 3 \cdot 6 = 18$ velocity at the end of third second
$v = 0 + 4 \cdot 6 = 24$ velocity at the end of fourth second
$v = 0 + 5 \cdot 6 = 30$ velocity at the end of fifth second
$v = 0 + 6 \cdot 6 = 36$ velocity at the end of sixth second
$v = 0 + 7 \cdot 6 = 42$ velocity at the end of seventh second

Hence, the velocity of the car at the end of seven seconds is 42 ft./sec. or 28 7/11 miles/hour.

Suppose that the car accelerates 4 feet per second per second. How much time would it take to change its velocity from 30 miles an hour to 40 10/11 miles an hour or from 44 ft./sec. to 60 ft./sec.

In this case 44 is the initial velocity and 60 the final.

$v = 44$ velocity at the start of the increase
$v = 44 + 4 = 48$ velocity at the end of first second

$v = 44 + 2 \cdot 4 = 52$ velocity at the end of second
$v = 44 + 3 \cdot 4 = 56$ velocity at the end of third second
$v = 44 + 4 \cdot 4 = 60$ velocity at the end of fourth second

From this we see that it would take 4 seconds.

If we plot velocity on the ordinate and time on the abscissa, we have a space curve for uniform acceleration of 4 ft./sec. The graph is a straight line.

Fig. 4

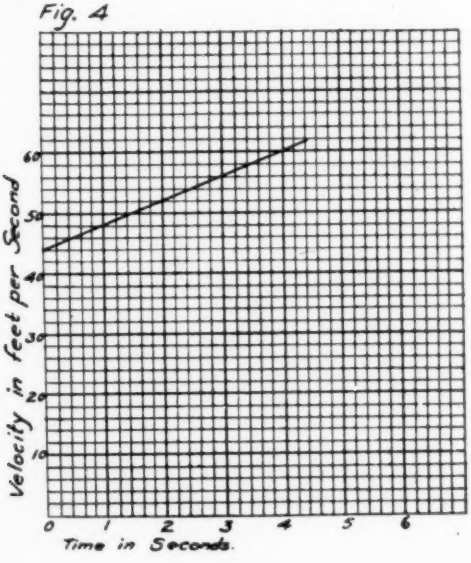


Figure 4

If v = final velocity, v_1 = initial velocity, a = acceleration and t = time, we have

$v = v_1$ velocity at the time of start
$v = v_1 + a$ velocity at end of first second
$v = v_1 + 2a$ velocity at end of second second
$v = v_1 + 3a$ velocity at end of third second
$v = v_1 + 4a$ velocity at end of fourth second

.....
.....
.....

$v = v_1$ at final ~~victory~~ velocity

This is true for the relation between the coefficient of a to the time that has elapsed is the same.

By the use of this formula let us see if we can find the time required for an automobile to slow down from 45 miles/hour to 30 miles/hour. The retardation, which is really negative acceleration, is 8 feet per second per second.

45 miles/hour = 66 feet/sec. = initial velocity

30 miles/hour = 44 feet/sec. = final velocity

8 feet per second per second = 8 ft./sec.^{2} = retardation = negative acceleration = $-a$.

$$v = v_1 + at \quad \left| \begin{array}{l} a = -8 \\ v_1 = 66 \\ v = 44 \\ t = ? \end{array} \right.$$

$$44 = 66 - 8t$$

$t = 4\frac{1}{2}$ sec. Ans.

Definitions: *Acceleration* is the time rate of change of velocity, or increase or decrease in the velocity per unit of time.

Uniform acceleration is a constant time rate of change of velocity, or the same increase or decrease in the velocity per unit of time.

PROBLEMS.

1. A train attains a velocity of 54 miles per hour in 4 1/2 minutes after starting from rest. Find the acceleration.

Ans. .293 ft./sec.^{2}

2. A ball thrown across the ice started with a velocity of 80 ft./sec. It was retarded by friction at the rate of 2 ft./sec.^{2} How long will it roll?

Ans. 40 sec.

3. A body whose acceleration is 12 ft./sec.^{2} attains a velocity of 1224 ft./sec. in 8 seconds. Find the initial velocity.

Ans. 1128 ft./sec.

4. What acceleration must a body have to obtain a velocity of 30 miles per hour in half a minute after starting from rest? What retardation would destroy this motion in ten seconds?

Ans. 1.467 ft./sec.^{2}

4.4 ft./sec.^{2}

THE PLACE OF THE HISTORY AND RECREATIONS OF MATHEMATICS IN TEACHING ALGEBRA AND GEOMETRY

By LAO G. SIMONS
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Longfellow[1], in the story of the schoolmaster, says, "There is something divine in the science of numbers. Like God, it holds the sea in the hollow of its hand. It measures the earth; it weighs the stars; it illumines the universe; it is law, it is order, it is beauty," and the master goes on to say, "and yet we imagine—that is, most of us—that its highest end and culminating point is bookkeeping by double entry. It is our way of teaching it that makes it so prosaic."

Among the means to be used in vitalizing the teaching of mathematics in the secondary schools are to be found the applications of the subject in its history and recreations.

Recommendations to this end are contained in the reports of committees appointed by organizations of mathematicians and of teachers[2]. The two most significant of these are cited. The International Commission on the Teaching of Mathematics says, "The candidate for a certificate to teach mathematics in our schools should be required to take—history of mathematics and its bearing on teaching." The report of the National Committee on Mathematical Requirements[3] says, "Teachers are advised to make themselves reasonably acquainted with the leading events in the history of mathematics. . . . They should use this material incidentally throughout their courses for the purpose of adding to the interest of the pupils by means of informal talks on the growth of mathematics and on the lives of the great makers of the science."

At the other extreme of recommendation is a personal experience related by a preacher heard some time ago. As a boy, he could not understand arithmetic, could not understand how ½ could be contained in ¼ when ½ is greater than ¼. A new teacher came to the village, one who interested himself in his

[1] Longfellow, H. W., "Hyperion—Kavanagh," ed. 1888, p. 301.
[2] Bureau of Education, Bulletin 1911, No. 16, p. 81.
[3] Bureau of Education, Bulletin 1921, No. 32, p. 22.

pupils as individuals. He found out that the boy enjoyed history, and thereupon loaned him a history of mathematics. The boy read this book with intense interest, and his whole attitude toward mathematics was completely changed. He came to enjoy it and to find it simple and easy. A connection had been made between what satisfied him and his mathematics in such a way as to develop genuine mathematical enthusiasm and ability.

The reasons for including the history and recreations of mathematics in secondary school work are associated with specific aims in teaching mathematics. The specific aims of the immediate lesson are the acquisition of facts and skill in processes, interest in and appreciation of algebra and geometry for their applications and for their own sake. Every possible legitimate stimulus must be sought to realize these aims. The field of history and recreations has been too little considered in this connection.

The history and recreations of mathematics awaken and maintain interest in the subject:

1. Through the presentation of the beginnings of a subject or the need of mathematics. Geometry, as its name signifies, had its beginning (traditionally) in the measuring of land in Egypt for taxation. The overflow of the Nile erased boundaries, and the land had to be re-measured to determine the taxes. Thales took these facts and tried to find an explanation for them, and so made a beginning in the science of geometry. Pupils may be told this, and then shown how he deduced other facts from them. They will then grasp the meaning of a proof more readily. New discoveries, such as the properties of angles in a semi-circle, can be made experimentally and explanation and proof sought. Or, again, the need of mathematics may be shown in connection with a proposition, such as the Pythagorean. The work of the Egyptian rope-stretchers who used the 3, 4, 5 triangle in orienting temples and of the Brahmin priests who used the 15, 36, 39 triangle in India will be an objective treatment that will show the natural source of the early geometrical facts. And the pupil will identify himself with these beginnings, and project himself into these situations.

2. Through an understanding of algebra and geometry as growing subjects, not as cut and dried rules and definitions,

propositions and demonstrations. As an illustration, an outline of the history of symbolism may be built up. This would be an aid in understanding the meanings of terms and operations. It may emphasize the fact that symbolism is independent of language, so that one would understand the symbols of a modern algebra written in Sanskrit or Chinese.

3. Through the consideration of a fact in history in relation to a process and its later development. As a class takes up a new topic like the transposition of terms in a simple equation, the meaning of the word "algebra" may be given to the pupils. The Arab Al Khowarazmi called his work "Al-jabr w'al muqabalah"[1] which were the names of the two operations that he considered it necessary to perform before his mechanical rule could be applied to the problem. The first operation (literally, to reduce a fracture) was to transpose all negative terms so that only positive ones would remain. Though the admission of negative numbers when they meant debts as opposed to assets, and related ideas, until the final stage in the understanding of negatives in Descartes's graphic representation, the necessity for any such operation ceased, although the name remained. Pupils may be made to appreciate that the other part of the title, since it meant collecting all similar terms, would have been a more significant one to retain. A deep and lasting impression of the process of transposition and the value of the negative number will result.

4. Through acquaintance with a chosen few great minds, Ahmes, Pythagoras, Euclid, Al Khowarazmi, Leonardo of Pisa, Descartes, as real human beings who were once youngsters[2] like themselves, growing into men, who played and worked like men today. This may be accomplished in connection with certain propositions in geometry or rules in algebra. The use of the five-pointed star as a badge by the Pythagorean brotherhood, with the letters for "health" in the points, and the significance of this may be given in connection with the construction of a pentagon. This would make a distinct appeal to the social instincts of the pupils. Thales and his measuring of the height

[1] Peman and Smith's translation of Fink's History of Mathematics, p. 88.
[2] Smith, David Eugene, "Number Stories of Long Ago."

of a pyramid may be used in connection with similar triangles. This particular fact has a use, moreover, in developing spatial imagination.

5. Through linking up mathematics with persons and places already familiar to the pupil, Haroun al Raschid and the Arabian Nights, Lewis Carroll (Dodgson) and Alice in Wonderland, Longfellow and Hindu problems, Alexandria, Athens, Bagdad, Pisa. Here are multiple and varied associations, and the more of such connections that we can have, the more will mathematics "permeate and enthrall" the pupil.

Further, just as the tension in an audience may be relieved and a certain set of the mind broken up so that the speaker can continue and produce the impression he wanted to make, so relaxation from a certain monotony inherent in formal algebra and geometry may be attained by means of an historical anecdote or incident. Along this same idea, it seems entirely legitimate to appear to simply entertain a class, as a reward for some special attainment, by reading a story like Stephen Leacock's "A, B, C"[1] in algebra or parts of a book like "Flatland"[2] in geometry. The latter may, indeed, have its uses in advancing ideas of space. Both of these books involve mathematical concepts, and could not be appreciated without a knowledge of mathematics. The pupil may be required to make a response by taking notes on these mathematical points. An account of a very original treatment[3] of magic squares shows a recreation used in a way related to the foregoing. The writer of the article tried it during the two weeks just before the Spring vacation. The formation of the algebraic squares gives considerable practice in addition, multiplication, and division. The writer states that the interest aroused by the magic squares not only carried the class through a trying period of the year but secured a mental alertness which transferred to the other work.

History and recreations can be employed in the acquisition of skill:

1. Through drill. There are instances which, although no claim of superiority to other means can be made for them, offer var-

[1] Leacock, Stephen, "Nonsense Novels."

[2] Abbott, Rev. E. A. "Flatland."

[3] McLaughlin, H. B., "Algebraic Magic Squares" in Mathematics Teacher, February, 1921.

iety of drill in certain topics. Algebraic problems and other material known by the pupils to come from the works of men with whom they have become familiar will give drill in an agreeable form. In the work of Longfellow[1] already referred to is found a collection of fascinating problems, both algebraic and geometric, which are taken from a translation of the Lilāvati of the Hindu mathematician Bhāskara. The book may be in the class-room, and the class put to work on the whole set to complete it within a certain time. This is drilling without coming out in the open about it, and is at the same time an appeal to the instinct of emulation and desire for approbation.

2. Necessary drill in one particular type of problem may be accomplished through a study of its history[2]. The problem of pursuit and the problem of the cisterns may be handled in this way. The problems would have to be provided by the teacher (and some stated in rhyme[3] may be included) but the pupils can get hold of variations of the type. These types of problems may disappear from the text-books in the near future, but at present, they are there and must be reckoned with. Indeed, a few such types can as safely be included as some of the so-called "real" problems of today. In the two illustrations given, we are creating situations in which habits can function.

3. The history and recreations of mathematics can be drawn upon for the correction of certain tendencies or mistakes. The pupil may not have realized the need of forming a habit because he has gotten along fairly well in his work without it.

In geometry, the careful drawing of diagrams is a habit to be formed. There are ways to accomplish this through an appeal to the aesthetic, in geometric design, at the outset of the course. This appeal does not always meet with the hoped-for response. Then the presentation of the fatal effects of inaccuracy through the geometrical fallacy is worth-while. The most telling one of these is the proof that a right angle equals an obtuse angle[4], a fallacy due to inaccurate drawing. These fallacies have the additional value of using geometrical demonstration.

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[1] Longfellow, H. W., loc. cit. pp. 303-305.
[2] Smith, David Eugene, "On the Origin of Certain Typical Problems." American Mathematical Monthly. February, 1917.
[3] Teachers College Record, November 1912, p. 65.
[4] Ball, W. W. R., "Mathematical Recreations and Problems," ed. 1911, p. 45.

In algebra, the double sign for the square root of an expression is easily forgotten, or rather, the sign is not considered at all because a pupil does not think of the convention that absense of sign indicates a positive quantity. A recreational appeal such as the following[1] may serve to supplement the algebraic reason, and is almost certain to produce the desired effect.

$9 + 5 = 2 \times 7$
Multiply both members by $9 - 5$
Then $9^2 - 5^2 = 2 \times 7 \times 9 - 2 \times 7 \times 5$
$\therefore 9^2 - 2 \times 9 \times 7 = 5^2 - 2 \times 5 \times 7$
Adding 7^2 to both members
$\therefore 9^2 - 2 \times 9 \times 7 + 7^2 = 5^2 - 2 \times 5 \times 7 + 7^2$
Taking the square root
$9 - 7 = 5 - 7$

Hence $9 = 5$, a result due to the failure to consider signs in taking the square root.

In both these cases, we are really overcoming habits through what is in a sense "negative illustration." Young folks like the "trick" element, too, and will master the details, with a certain sense of power, in order to try the trick on some one else.

The history and recreations of mathematics can be drawn upon for illustrations. Such a use has already been pointed out in arousing interest and overcoming habits. Here, the term is used in the strict sense "to make clear." In using the check of arbitrary values in division of a polynomial by a polynomial in algebra, pupils are told that the divisor must not equal zero, and they may be told further that, if they continue the study of mathematics, they will learn the reason. A better method is to use an algebraic recreation such as the following[2]:

Let $a = b$
Then $ab = b^2 = a^2$
Hence $ab - b^2 = a^2 - b^2$
and $b(a - b) = (a + b)\ (a - b)$
$\therefore b = a + b$
or $b = 2b$
and $1 = 2$

[1] Teachers College Record, loc. cit. p. 109.
[2] Teachers College Record, loc. p. 108.

since $a = b$, $a - b = 0$, and in the fifth step the divisor zero was used. Hence, division by zero cannot come under the same rules as division by other numbers.

In order to use all of this material in the class-room, the teacher must have an historical background through a thorough study of the history of mathematics. Such familiarity with the history enables a teacher to make incidental references whenever occasion arises, although the plan for each day's lesson should include preparation to this end. Such a familiarity enables the teacher to see that mathematics as a science and mathematics as an art are inseparably bound together.[1] The teacher is better able to appreciate the difficulties of his pupils, if he knows the difficulties of the race in developing the concepts to be received in mathematics. History will aid the teacher through showing the beginning of algebra and geometry and how and when certain concepts arose. This will be a guide[2] in the presentation of the subject and in the point at which it shall be introduced. Against this view, however, may be placed the charge that to psychologists is due the retardation in the development of certain mathematical notions.[3]

The co-operation of the members of the class can be secured in presenting some of the subject-matter. If this is to be the procedure, very explicit references to attractive and readable books should be made. Indeed, there should be a shelf of books and articles from magazines in the class-room.

The incidental references of the teacher may be supplemented by the use of photographs, an admirable series of which is now attainable.[4] It may also be supplemented by lantern-slide talks, for slides are available.[5] Probably only one such talk on algebra and one on geometry would be feasible, and these should be given when the subject is well advanced, and with the purpose of outlining very definitely one phase of its growth.

The establishment of a museum appeals to the instinct to collect and the instinct to construct, to make things. Material of an historical nature connected with secondary mathematics is

[1] Nunn, T. P., "The Teaching of Algebra," p. 16.
[2] Zilliacus, Laurin, "The History of Mathematics in Elementary Instruction," American Mathematical Monthly, February 1920.
[3] Keyser, C. J., "Mathematical Philosophy," pp. 407-410.
[4] Open Court Publishing Company, Chicago.
[5] Teachers College, New York City.

meager and difficult to obtain, but there are possibilities along the line of constructing models, such as a "book" of geometry or a crude instrument to show how Thales probably measured angles. This museum would include objects gathered in connection with applications outside of the history and recreations.

There is no intent in this paper to make the history and recreations so prominent that the chief emphasis seems to be along that line. They constitute one of the methods by which the work can be made interesting, significant, alive. Through these methods, the pupil will come to realize that he is studying subjects that were centuries in developing and that are to-day having wider applications than ever before. He will be spurred to greater effort and finer work. For the teacher, to give his pupils joy in mathematics, for the sake of its applications, for the sake of its part in the progress of the world, and, above all, for the sake of the subject itself is satisfaction beyond all material reward.

A STUDY OF THE CULTIVATION OF SPACE IMAGERY IN SOLID GEOMETRY THROUGH THE USE OF MODELS

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Goethe, in his Sprüche in Prosa, makes the following statement: "Alles Gescheite ist schon gedacht worden; man muss nur versuchen es noch einmal zu denken." Thus it is with a feeling of certainty that nothing distinctly new will be offered by the present writer and yet the presentation of the facts in the case may be sufficiently different to cause one to pause for a few minutes and reconsider some of the fundamental truths.

After ten years of experience in the teaching of solid geometry in the high school the writer is convinced that the secondary school student needs *definite* training in the cultivation of space imagery. Too much is taken for granted in this field by the average teacher and few resort to any *definite* method of overcoming the difficulty. Perhaps the pronounced neglect in this training which the pupil really needs for a full appreciation of the study of solid geometry is in part due to the fact that the literature in this field of cultivating space imagery is rather scattered and not easily accessible. A gem from a book on the teaching of mathematics, written a half century ago by a distinguished pedagoge of Columbia College, may throw some light on method in 1873. In his suggestions for those who teach geometry he has this to offer. "Be sure that your pupils have a clear apprehension of space, and of the notion that geometry is conversant about space only." The title of the book is "The Nature and Utility of Mathematics, with the best Methods of Instruction explained and illustrated." The author's advice just quoted is to the point but he forgot to "explain and illustrate" the method of procedure.

Geometry, and more particularly solid geometry, is made up of two distinct parts, namely: (1) space perception, and (2) scientific analysis. It is the purpose of this paper to discuss the first part, and that primarily as cultivated through the use of models.

In his book on "The Teaching of Mathematics in the Secondary Schools," Schultze says: "Under the conditions that prevail in our secondary schools, and with the time usually allowed for the subject, it seems that solid geometry cannot be made a subject of discovery and a discipline for training the mind to quite the same extent as plane geometry. Exercises in solid geometry of purely demonstrative character are comparatively hard to construct and often more difficult to solve than those in plane geometry; the amount of book matter to be studied is relatively large, and the danger of making students learn by heart many proofs which they would never discover themselves, is greater than in plane geometry. On the other hand the study of solid geometry strengthens the student's space imagination and his power to image space configurations, and it gives him an understanding for drawings that represent spacial objects.

"Altogether it seems that the utilitarian advantages are somewhat greater, but the purely cultural advantages somewhat smaller, than in plane geometry. In accordance with this view, textbooks devote a large part of their space, and schools a large part of their time, to mensuration and to the theorems that, directly or indirectly, lead to mensuration.

"With such restrictions the study of solid geometry will not offer great difficulty to the student. It may require a little more time and a little more study, but it does not require more intelligence than does plane geometry.

"One difficulty, however, against which we must guard and which we must overcome at the very start is the inability of some students to understand diagrams of solids. There are students who are able to reason logically, but who cannot imagine clearly the spacial forms which the diagrams represent. There are two ways of overcoming this difficulty, namely the use of models and rational methods of drawing.

"The function of the model is to help the student in the beginning to an understanding of solid figures in general, and to make clear to him, later on, difficult drawings which otherwise he would not understand. The model should, however, not be used to supplant the drawing. As soon as the student is able to understand the drawings, the models should be discarded or re-

served for the most difficult cases only. Otherwise the student will lose one of the main benefits of the study, viz., the development of his space imagination and of his faculty to understand diagrams of solids. Matters difficult to depict, however, such as the regular polyedrons, the distinction between right and rectangular parallelopiped, etc., should be explained by the use of models. Another reason why models should be used only sparingly is that, as a rule, they cannot be shown to a large class as readily as a blackboard diagram, but must be explained to the students individually or to small groups of students, thus sacrificing a great deal of time.

"As a rule the simple, inexpensive model of paper, strings, wire, etc., serves its purpose just as well as the most expensive one. A triangular prism, cut out of a potatoe, and divided into three equivalent pyramids is just as instructive as the most expensive model. Many of the propositions relating to lines and planes may be illustrated by a couple of pencils, a book, a piece of paper.

"As far as the making of models by the students is concerned, there can be no doubt that in many cases the student's understanding will be improved thereby. A student who makes a regular icosaedron, or dodecaedron out of cardboard *has undoubtedly a much clearer notion* of these solids than he had before. On the other hand, it is doubtful whether the expenditure of a great deal of the student's time for the making of many or elaborate models is justified. The time may be well invested as far as manual training is concerned, but not as far as it relates to mathematical reasoning. Especially the custom of exhibiting mathematical work and models made by students must be strongly condemned. Such exhibitions not only raise an utterly wrong standard for the measurement of the result of mathematical work, but they are often like many educational exhibits—deceptive and misleading."

The long quotation of Schultze's is given to bring to our attention a detailed point of view of a leader in mathematics who has had a great influence in molding the content of our secondary school courses. He was for several years a member of the College Entrance Examination Board at a time when our high schools were primarily interested in preparing students for col-

lege. Since the publication of his book (1912) little has been written in the field of solid geometry, and more especially on the use, or abuse, of models.

D. E. Smith, in 1911, makes the following suggestion: "An extensive use of models is, however, unwise. The pupil must learn very early how to visualize a solid from a flat outline picture, just as a builder or a mechanic learns to read his working drawings. To have a model for each proposition, or even to have a photograph or a stereoscopic picture, is a very poor educational policy. A textbook may properly illustrate a few propositions by photographic aids, but after that the pupil should use the kind of figures that he must meet in his mathematical work. A child should not be kept in a perambulator all his life,—he must learn to walk if he is to be strong and grow to maturity; and it is so with a pupil in the use of models in solid geometry."

Again we have a very restricted application of the use of models in solid geometry by an author of textbooks widely used in this country. From the point of view of the specialist in mathematics, as illustrated by the two cases cited, it is evident that he would be inclined to discourage the use of models because it might cause the pupil to stumble in his later work in *scientific analysis*. But what about the cultivation of space imagery? Is the student to acquire this phase of geometry as best he may? Do a few "setting-up exercises" establish a proper introduction and regard for *scientific analysis* in the study of solid geometry? Let us look at the matter from a psychological point of view.

Wundt, in his Outlines of Psychology states "that all spatial ideas are arrangements either of *tactual* or of *visual* sensations. Indirectly, through the connection of other sensations with either tactual or visual ideas, the spatial relation may be carried over to other sensations through association of ideas. In the cases of touch and sight, it is obvious that the extended surface of the peripheral sense-organs, and their equipment with organs of movement, which render possible a varying location of the impressions in regard to the ideating subject, are both favorable conditions for an extensive, spatial arrangement of the sensations. The tactual sense is the earlier of the two here in question, for it appears earlier in the development of organisms and

shows the structural relations in much coarser, but for that reason in many respects much plainer form than does the more delicately organized visual organ. Still it is to be noted that where vision is present, the spacial ideas from touch are greatly influenced by the ideas from sight, because of the higher development of vision.

"The perception of absolute distance and of difference in distance, is greatly improved by practice. Children are generally inclined to localize very distant objects in the immediate neighborhood; they grasp at the moon. It is of importance for the development of this discrimination between far and near under the natural conditions of vision, that not mere isolated points be represented, but *extended three-dimentional objects.*"

Thus it seems to be distinctly necessary to give considerable time to practice in the use of the *tactual* and *visual* organs in order that we may develop proper spatial ideas. The young student in solid geometry, many of whom are not more than fifteen or sixteen years of age, needs *definite* training in this regard and one of the best methods, and to him the most interesting—for we rarely "get thin" to disciplined drill but rather to music,— is the construction of paper models. Not necessarily models of the propositions that have come down to us from the father of geometry and that some how or other must be proved that we may say with impunity— "I have followed the lean and narrow path and have not set foot upon a royal road;" but models that are of interest to the young mind and which will, whether he knows it or not, develop spatial ideas. Models, as illustrated in the figures on the next page, showing many kinds, some of which are of interest because they are familiar, others because they are difficult, and still others because they are beautiful.

The writer has used the model method for some eight years in Pennsylvania and Illinois and is convinced that the young pupil has a greater appreciation of solid geometry in all of its phases because he has built these mathematical blocks. They are as crystals to him through which he can see to better advantage the meaning of scientific analysis. As Branford notes in his Betrachtungen über Mathematishe Erziehung: "Begriffe oder

Ideen sind bloss leere Worte in jedem Geiste in welchem sie nicht eine grosse Gruppe wohlgeordneter Sensationen oder Sinneseindrucke wachrufen."

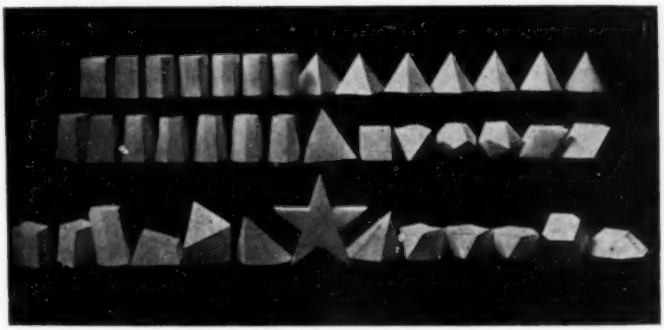

Figure 1.—41 models made by the students of Mr. Schreiber's solid geometry class, second semester, 1922. Top row shows regular prisms and pyramids. Middle row shows frustums of regular pyramids and regular polyhedrons. Bottom row shows some interesting "original" polyhedrons. All models have a volume of 100 cubic centimeters. Each student makes 10 selected models.

Let us look into the life history of one of these models to get a clearer mental picture of the method of procedure. An exercise reads: Required to construct a cube whose volume is one hundred cubic centimeters. (The writer believes in *using* the metric system in mathematical work rather than *teaching about it*). To determine the length of the edge of the cube requires the solution of the equation $e^3 = 100$. Solving for e we obtain $e = \sqrt[3]{100}$ or 4.642, approximately. The approximate answer raises the question as to how the student is to extract the cube root of 100. If the class is not familiar with the use of logarithms, spend several recitations in explaining their use and you will be surprised how quickly this powerful computing tool is successfully operated by the average pupil. He does not need to *know all about* logarithms to be able to *use* logarithms. If he has access to a table of powers and roots of numbers (which table ought to be in every high school textbook in mathematics, as well as a table of logarithms) he may use it to find the cube root of 100. The approximate result is required to three places of decimals so that the nearest hundredth of a centimeter, or the

tenth of a millimeter, in length is determined, which is the "least count" in actual drawing. The student must be firmly impressed with the fact that to draw with such accuracy re-

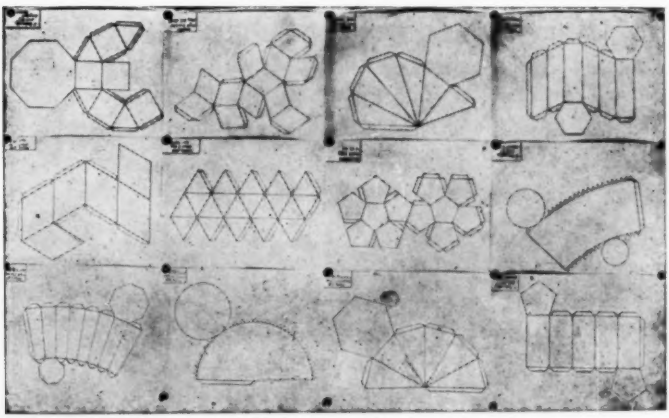

Figure 2.—Patterns of 12 models on tracing paper made by students of Mr. Schreiber's solid geometry class, second semester, 1922. A good drawing bringeth forth a good model.

quires greater skill than he has been accustomed to, the error allowed being $\pm$. 1mm. To insure such accuracy the diagrams are drawn on a good quality of tracing paper (Eugene Dietzgen Co. No. 188). The tracing paper allows blueprints to be made from the original diagram, a feature which at times can be used to good advantage. Furthermore, the easiest method to check the diagram of a pupil is to superimpose the instructor's drawing upon the pupil's and if they coincide, the work is accepted. This method is possible because the paper is semi-transparent.

After the diagram has been presented and approved the pupil uses it as his pattern and the real construction of the cube is begun. It is made out of a good drawing paper (Dietzgen No. 5). The pattern is placed upon the blank drawing paper and by means of a well sharpened pencil (4 or 6H) the necessary points are transferred by a slight downward pressure. The points are then joined by the required lines and thus the entire diagram is transferred. Perhaps you are wondering why a good diagram is made on the tracing paper only to be transferred to the drawing paper. The reason is simple. If the first model is not ac-

cepted, as is often the case with beginners, the time and effort required to make the original diagram is not lost. It is only a matter of using the pattern a second time. With the aid of a pair of scissors the diagram is cut out along the boundary lines and where folding along a line is necessary a knife is used to make a shallow cut. It has been found that the use of flaps in joining two edges together is better and makes a neater model than the use of gummed paper. Either glue or library paste may be employed as an adhesive but the paste has a tendency to dry out in time and the model comes apart. The cube is now complete and if satisfactory is accepted and the pupil assigned a new construction exercise.

It is plainly evident if one has followed the foregoing procedure that the pupil gets a very detailed and *definite* training in the use of his tactual as well as visual sense-organs which the writer confesses is a legitimate part of the training necessary to fully appreciate the logic of Euclid.

CONSTRUCTION EXERCISES IN SOLID GEOMETRY

(A) PRISMS

Note: In the following seven exercises the letters are defined thus:

A—The area of the base.
h—The altitude of the prism.
R—The radius of the circle circumscribing the base.
s—The side of the polygon forming the base.
V—The volume of the prism.

1. (*A*) Construct a regular prism whose base is a triangle; *A* is 15 square centimeters, *V* is 100 cubic centimeters. Find *h*, *R* and *s*.

2. (*A*) Construct a regular prism whose base is a square; *A* is 15 sq. cm., *V* is 100 cc. Find *h*, *R* and *s*.

3. (*A*) Construct a regular prism whose base is a pentagon; *A* is 15 sq. cm., *V* is 100 cc. Find *h*, *R* and *s*.

4. (*A*) Construct a regular prism whose base is a hexagon; *A* is 15 sq. cm., *V* is 100 cc. Find *h*, *R* and *s*.

5. (*A*) Construct a regular prism whose base is a heptagon; *A* is 15 sq. cm., *V* is 100 cc. Find *h*, *R* and *s*.

6. (*A*) Construct a regular prism whose base is an octagon; *A* is 15 sq. cm., *V* is 100 cc. Find *h*, *R* and *s*.

7. (*A*) Construct a right circular cylinder; *A* is 15 sq. cm., *V* is 100 cc. Find *h*, *R* and *c* (circumference of the base.)

(*B*) PYRAMIDS

Note: In the following seven exercises, the letters are defined as under Prisms, with additional letter—l, the lateral edge of the pyramid.

8. (*B*) Construct a regular pyramid whose base is a triangle; *A* is 45 square centimeters; *V* is 100 cubic centimeters. Find *h*, *l*, *R* and *s*.

9. (*B*) Construct a regular pyramid whose base is a square; *A* is 45 sq. cm., *V* is 100 cc. Find *h*, *l*, *R* and *s*.

10. (*B*) Construct a regular pyramid whose base is a pentagon; *A* is 45 sq. cm., *V* is 100 cc. Find *h*, *l*, *R* and *s*.

11. (*B*) Construct a regular pyramid whose base is a hexagon; *A* is 45 sq. cm., *V* is 100 cc. Find *h*, *l*, *R* and *s*.

12. (*B*) Construct a regular pyramid whose base is a heptagon; *A* is 45 sq. cm., *V* is 100 cc. Find *h*, *l*, *R* and *s*.

13. (*B*) Construct a regular pyramid whose base is an octagon; *A* is 45 sq. cm., *V* is 100 cc. Find *h*, *l*, *R* and *s*.

14. (*B*) Construct a right circular cone: *A* is 45 sq. cm., *V* is 100 cc. Find *h*, *l*, *R*, *c* (circumference of base) and *d* (difference between 3.1416 ⟩ and *c*.

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SOME ASPECTS OF CORRELATION THEORY.[1]

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1. *THE MEANING OF CORRELATION.* The theory of correlation seeks to devise methods of measuring the extent to which two characteristics measured in a group of individuals are so related that there is a tendency for changes in the size of the one to depend upon changes in the size of the other. For example, in the realm of biological phenomena one might seek to determine the degree of correlation between height and weight in a group of individuals; in the realm of mental phenomena one might seek the correlation between algebraic ability and geometrical ability in the group, assuming that adequate measures of these abilities exist. Let the magnitudes of the two characteristics for the first individual be x_1 and y_1, for the second individual x_2 and y_2, and in general for the rth individual x_r and y_r. In the examples mentioned above x could be the height of an individual and y his weight; or x could be the measure of the algebraic ability of the individual and y the measure of his geometrical ability. If we employ rectangular coordinates, the magnitudes of the two characteristics for an individual determine a point in the plane. When the points for all the members of the "population" have thus been plotted we have what is known as a *scatter diagram.*

When the group to be studied is large it is convenient to replace the scatter diagram by a correlation table. The plane is divided into squares of any convenient size, and all the points of the scatter diagram falling within any one square are thought of as being concentrated at the center of this square. A number is then written within the square to designate the number of points at its center. If, at the center of each square a perpendicular to the XY-plane is erected equal in length to the number written within the square, the upper ends of these lines will all lie on the *correlation surface* of the data.

In essence the problem of measuring correlation is the problem of determining the extent to which the points of the scatter

[1] Presented before the Illinois Section of the Mathematical Association of America at Rockford College, Rockford, April 28, 1922.

diagram tend to concentrate about some curve fitted among them in such a way as to represent their general trend. In other words the object is to measure the extent to which the size of the one characteristic tends to become a single valued function of the size of the other.

2. *COEFFICIENT OF CORRELATION.* The simplest function the data might approximate is represented graphically by a straight line, and when we attempt to measure the approach to such a linear relationship we are led to a study of the Pearson Product-moment coefficient, a simple form of which is given by the formula[1]

$$r = \frac{\Sigma(xy)}{\sqrt{\Sigma x^2 \ \Sigma y^2}} \tag{1}$$

where the summation extends over all the points of the scatter diagram, and where the x's and y's are measured from their respective arithmetic means. This formula may be modified by introducing the standard deviations σ_x of the x's and the σ_y of the y's. These are defined by the equations

$$\sigma_x^2 = \frac{\Sigma x^2}{N} \text{ and } \sigma_y^2 = \frac{\Sigma y^2}{N} \tag{2}$$

where N equals the number of individuals in the population whose coefficient of correlation is to be calculated. Using these equations in (1) we get

$$r = \frac{\Sigma(xy)}{N\sigma_x\sigma_y}. \tag{3}$$

The standard deviations are measures of "dispersion"; they tell us the range of the distribution. Thus, the smaller the quantity σ_x is, the more closely do the x's concentrate about their arithmetic mean.

The properties of r can be studied by fitting a line $Y = m_1x + b$ to the scatter diagram in such a way as to make the sum of the squares of the vertical distances from the points to the line a minimum. The square of a vertical distance from one such point (x, y) to the line is $(Y - y)^2 = (m_1x + b - y)^2$.

[1] For a Discussion of practical methods of computing "r" see Rugg, H. O., "Statistical Methods Applied to Education," Chapter IX.

The line must be fitted so that

$$\Sigma(Y-y)^2=\Sigma(m_1x+b-y)^2=f(m_1,b) \qquad (4)$$

regarded as a function of m_1 and b shall be a minimum. The condition for a minimum is that the partial derivatives

$$\frac{\delta f}{\delta b}=0 \text{ and } \frac{\delta f}{\delta m_1}=0.$$

The first of these gives

$$m_1\Sigma x+Nb-\Sigma y=O.$$

Since measurements are taken from the mean for both x and y, we have $\Sigma x=O$ and $\Sigma y=O$, hence $b=O$. Simplifying (4) by using this last fact, and differentiating with respect to m_1 we have

$$\Sigma x\ (m_1x-y)=O.$$

$$\text{Therefore } m_1=\frac{\Sigma(xy)}{\Sigma x^2}=\frac{\sigma_y}{\sigma_x}r.$$

The last part follows from the middle after multiplying numerator and denominator by $\sqrt{\Sigma y^2}$ and using (1) and (2). Thus the line of best fit (the regression line of y on x) is

$$Y=\frac{\sigma_y}{\sigma_x}\,r\,x.$$

Using this value of Y in (4) we have

$$\Sigma\,(Y-y)^2=\Sigma\,(\frac{\sigma_y}{\sigma_x}rx-y)^2=$$

$$\frac{\sigma_y^2}{\sigma_x^2}r^2\,\Sigma\,x^2-2\,r\frac{\sigma_y}{\sigma_x}\Sigma\,(xy)\ +\ \Sigma\,y^2$$

Substituting values for Σx^2, Σy^2, and$\Sigma\ (xy)$ from (2) and (3) we have, after dividing through by N

$$\frac{\Sigma\,(Y-y)^2}{N}=\sigma_y^2(1-r^2\,). \qquad (5)$$

From (5) some of the properties of r may be inferred. Evidently $|r|\leqq 1$ since the left member and σ_y^2 are both positive. If the points all lie exactly on the regression line, the the left member of (5) vanishes and $r=\pm 1$ giving perfect correlation. The larger r is, the smaller the left member will be, and the greater the concentration of the points about the re-

gression line. If $r = O$, the regression line is horizontal, and there is no linear correlation.

By fitting a line $X = m_2 y + b$ to the points of the scatter diagram in such a way that the sum of the squares of the horizontal distances from the points to the line shall be a minimum, it is possible to deduce a second regression line (the regression line of x on y) whose equation is

$$X = \frac{\sigma_x}{\sigma_y} ry.$$

This regression line will coincide with the one derived above if, and only if, $r = \pm 1$. From the equations of the regression lines it is evident that if $r > O$, an increase in the one variate tends to accompany an increase in the other; if $r < O$, an increase in the one will be accompanied by a decrease in the other. Since $r = \sqrt{m_1 m_2}$, the coefficient of correlation summarizes in a single number the dependence of y on x and of x on y.

One very frequently hears a particular value of r referred to as indicating a "high" correlation or a "low" correlation. What constitutes "high" correlation and "low" correlation is largely an arbitrary matter depending on current practice and one's own experience or point of view[1]. While such criteria are of value in interpreting correlation coefficients, their empirical nature must make them rather unsatisfactory to the mathematically inclined. A deeper insight into the meaning of the size of a correlation coefficient is given by certain theorems of Professor Rietz who has developed correlation theory from the theory of pure chance by means of certain "urn schemata"[2]. One of his theorems reads as follows: *An urn containing white and black balls is so maintained that in drawing a ball the probability of getting a white ball is a constant p and that of getting a black ball is q = 1-p. The first drawing of a pair is to consist of s balls taken one at a time from the urn. The second drawing is to consist of s balls of which t are taken at random from the s first drawn, and s-t are drawn one at a time from the urn. Then the regression is linear, and the coefficient of correlation between the number of white balls in the first and sec-*

[1] Rugg, Op. cit., p. 256.

[2] Rietz, H. L., "Urn Schemata as a Basis for the Development of Correlation Theory, Annals of Math." Vol. XXI, pp. 306-322.

ond drawings of a pair is t/s, when the frequencies are a set of a priori most probable frequencies.

We see that a value of $r = .7$, for example, might be obtained in accordance with the theorem just quoted by letting the first drawing of a pair consist of 10 balls taken one at a time from the urn, and the second of 10 balls 7 of which are taken at random from the first drawing and the remaining 3 from the urn. Then, if the coefficient of correlation between the number of white balls in the first and second drawings is calculated for N pairs of drawings, as N increases indefinitely r will approach the value .7. Thus do we gain an idea of the extent to which common factors present in pairs of variates may influence the degree of correlation between them.

3. *CORRELATION RATIOS.* When the trend of the data is not approximately linear the use of r is misleading, and some more general measure of correlation must be sought. In place of the straight line $Y = m_1 x$ used above some more general function must be chosen to represent the trend of the data. Let this function $Y_x = f(x)$ be defined for a given value of x as the mean value of the y's of all those points of the scatter diagram which have that value of x. Algebraically $Y_x = \frac{\Sigma y_x}{n_x}$ where n_x is the number of points in the x — array of y's, and the summation is over all the y's of the array. The concentration of the points of the column about their mean may be measured in terms of their standard deviation which may be denoted by σ_{ay}. That is, $\sigma_{ay} = \frac{\Sigma (Y_x - y_x)^2}{n_x}$, where the summation is over all the y's in the array or column. Now, the concentration of the points of the entire table about the curve through the means of the columns may be measured by finding the mean value of all such expressions σ_{ay}^2 for all the columns of the table. But since there are more points in some colums than in others it will be convenient to weight the σ_{ay}^2 for each column by multiplying it by the number of points in the column. In this way we arrive

at an equation analagous to (5) which may be regarded as defining π_{yx}, the correlation ratio of y on x;

$$\frac{\Sigma(n_x\sigma_{ay}^2)}{N} = \sigma_y^2(1 - \pi_{yx}^2) \qquad (6)$$

Here the summation extends over all the columns of the table, and N is the number of individuals for which the correlation is to be measured. If the means of the columns all lie on a straight line (6) evidently reduces to (5) and the regression is said to be linear.

If only one y corresponds to a given value of x (and it is the approach to single valued functionality of any form that π_{yx} measures[1]) each $\sigma_{ay} = O$, thus making the left member of (6) equal to zero. Hence $\pi_{yx} = 1$, and the correlation is perfect. If $\pi_{yx} = O$, the right member of (6) equals σ_y^2, and a moment's reflection on the meaning of (6) tells us that in this case the points of the scatter diagram concentrate no more closely about the curve through the means of the columns than they do about a horizontal straight line drawn through the means of all the y's of the table. In this case there can be no correlation in the sense in which we are using the term. Evidently the extent to which π_{yx} differs from zero measures the closeness of the correlation. For computational purposes it is easy to deduce from (6) the expression

$$\pi_{yx} = \frac{\sqrt{\frac{\Sigma n_x Y_x^2}{N} - \bar{y}^2}}{\sigma_y}$$

where $\bar{y}$ is the mean of all the y's of the entire table.

Besides π_{yx} the correlation ratio of y on x, there exists for the same data a second ratio π_{xy} which measures the extent to which the points concentrate about the means of the horizontal arrays of x's. In this second correlation ratio y is regarded as the independent variable.

The correlation ratio will always be greater than r except when the regression is linear. In the latter case, as we pointed

[1] Cf. Huntington, E. V., "Mathematics and Statistics, with an Elementary Account of the Correlation Coefficient and the Correlation Ratio, American Math. Monthly," December 1919, p. 481.

out above, these two measures of correlation become identical. The quantity $\lambda = \pi^2 - r^2$ is taken as the measure of the non-linearity of the regression, and it is by finding an expression for the probable error of λ that Blakeman[1] has provided a means for deciding whether the coefficient of correlation or the correlation ratio should be used as the measure of correlation.

How unsuitable r is as a measure of correlation in case the regression is non-linear may be illustrated by an artificial case. Let r be calculated for all points on the arc of the curve $y = \cos x$ from $x = -\pi/2$ to $x = \pi/2$. Since y is a single valued function of x, $\pi_{yx} = 1$. However, it follows from (1) that $r = 0$ since $\Sigma(xy) = \int_{-\pi/2}^{\pi/2} x \cos x \, dx = 0$. It is evident that $r = 0$ indicates absence of *linear correlation only* and not absence of correlation in general.

4. *PROBABLE ERRORS.* For the sake of brevity the discussion of the probable errors will be confined to the case of r, but the general implications of the remarks will apply with equal force to the probable errors of the other frequency constants, such as σ, π, λ, etc.

In most correlation work the population regarding which a conclusion is to be drawn is so large that it is difficult if not impossible to deal statistically with all of it. In such cases it is necessary to gain information regarding the total population by the study of a random sample of workable size drawn from it.

It is first necessary to define what is meant by a random sample. Suppose that for each member of a large population of N individuals there corresponds a card or ticket bearing the numerical measures for that individual of the two characteristics whose coefficient of correlation is to be calculated. Let all these cards be placed in a bag and let n of them (n being small relative to N) be drawn at random from the bag. These n individuals will then constitute a random sample for which r can be calculated. The question is: To what extent will the value of r for the sample approximate the value of r for the total population? It is clear that if the n cards were returned to the bag, and a second random sample of the same size drawn

[1] Blakeman, J., "On Tests for Linearity of Regression in Frequency Distributions, Biometrika," Vol. IV, Part III, pp. 332-350.

and r again calculated, this second value would not be identical with the first owing to the operation of chance. These variations in r for the sample from its true value for the total population are known as the errors of random sampling.

Let the drawings be continued until all possible samples of n individuals are taken, and let the coefficient of correlation between the two traits for each sample be determined. Denote these by $r_1, r_2, r_3, \ldots$. If the mean value of all of these r's is given by $\bar{r}$, the error of sampling for a particular r is given by $\triangle r = r - \bar{r}$. The extent to which all the r's concentrate about their mean may be measured in the usual way by means of their standard deviation. That is, square $\triangle r$ for every sample, sum all such squares, and divide by the number of samples. The square root of the result is the desired quantity, and it is symbolized in Pearsonian notation by Σ_r. The problem is to find a formula for this quantity in terms of the constants of the frequency distribution. By means of some rather intricate mathematics it is possible to obtain the desired form[1]:

$$\Sigma_r = \frac{1 - r^2}{\sqrt{n}} \tag{7}$$

It should be noted that an assumption underlying the mathematical derivation of this form is that the correlation surface of the original data is normal.[2] The general formula for the standard error of r when the distribution is of any form is a much more complicated expression involving the fourth and lower moments of the distribution.[3] However, statistical experience would seem to justify the use of the short form (7) even when the underlying assumption as to the normality of the distribution is far from realized. For the details of the nature of the argument one should refer to Pearson's discussion of the probable error of the correlation ratio[4].

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[1] Pearson, K., and Filon, L. N. G., "On the Probable Errors of the Frequency Constants, and on the Influence of Random Selection on Variation and Correlation. Phil. Trans. Roy. Soc. of London," Series A, Vol. 191, pp. 229-245.

[2] For definition and properties of a normal correlation surface see Yule, G. U., "An Introduction to the Theory of Statistics," 1916, Chapter XVI.

[3] Pearson, K., "On the General Theory of Skey Correlation and Nonlinear Regression. Drapers' Company Research Memoirs," 1905, footnote, p 20.

[4] Ibid., pp. 31-45.

It is conventional in statistical practice to multiply the standard deviation by .67449 thus obtaining the probable error of r:

$$E_r = .67449 \frac{1 - r^2}{\sqrt{n}} \qquad (8)$$

This simply means that if the r's for all random samples of a given size are assumed to be distributed according to the normal curve of error, 50% of these r's will lie within the range $r \pm E_r$ and 50% without this range. Under the same assumption as to the normality of the distribution of the r's, reference to a table of values of the probability integral will show that the odds are about 142 to 1 that the value of r in a sample will fall within the range $r \pm 4E$. If a value of r found in a sample exceeds $4E_r$ it is regarded as being "significant," i. e., indicative of correlation in the total population.

When n, the number of individuals in the sample, is small formulae (7) and (8) are inapplicable. One reason for this is that when n is small the assumption that errors in sampling are distributed normally is not likely to be realized. It is only when the errors in the frequency constants such as r are so small that those terms, in a certain power series, which involve the cubes and higher powers of these errors may be neglected that we may expect the distribution of errors in these constants to approximate normality. These errors in the frequency constants will be small enough to permit the dropping of the terms involving their higher powers only when n is sufficiently large.

Again it must be remembered that there is only the value of r in the sample to use in (8), and the smaller n is, the more r may deviate from its true value, and hence give misleading values to E_r. Indeed, in deriving some of the theorems which lie at the basis of the theory of random sampling it is necessary to replace in the formulae certain *a priori* most probable frequencies by the actual value of the frequency in the sample, and this is a safe procedure only when the size of the sample is great enough to give stability to the frequencies in question.[1] It is clear, therefore, that when n is small, the use of formula (8) and the other formulae for probable errors is dangerous and may result in wrong conclusions.

[1] Cf. Elderton, W. P., "Frequency Curves and Correlation," p. 134.

Finally, it must be remembered that many of the samples one is compelled to deal with in practice are by no means random samples in the sense defined above. The formulae (7) and (8) cannot be expected to cover fluctuations in r due to errors of observation, nor can they be safely applied when the sample is biassed, i. e., not truly representative of the data in the sense that a random sample is.

5. EFFECT OF HETEROGENEITY OF MATERIAL ON THE CORRELATION COEFFICIENT. In Yule the following theorem is given[1]: *If x and y are uncorrelated in each of two records, they will nevertheless exhibit some correlation when the two records are mingled, unless the mean value of x in the second record is identical with that in the first record, or the mean value of y in the second record is identical with that in the first record, or both.*

Such apparant correlation which arises solely through the mixing of two records has sometimes been called "spurious correlation". The above theorem does not cover all the possibilities, however, for correlation coefficients may have their values *lowered* as well as raised by mixing heterogeneous material. The following artificial, numerical example will probably enforce this point as well as a more lengthy algebraic discussion.

Let there be given two records as follows:

Record I.

x	0	—1	—2	—3	—4
y	2	1	0	—1	—2

Record II.

x	0	1	2	3	4
y	—2	—1	0	1	2

For Record I the correlation coefficient r_1 between x and y is equal to 1; similarly for Record II, $r_2 = 1$. This follows at once since in each case all the points of the record lie on a straight line. For Records I and II combined the value of r as calculated by formula (1) is

$$r = \frac{20}{\sqrt{1200}} = .577+$$

[1] Op. cit., pp. 218-219.

The principles just mentioned must always be taken into consideration in practical work. They have frequently been overlooked in educational applications when widely differing classroom groups have been combined before calculating the coefficient of correlation. Two groups may safely be combined only when they may be regarded as random samples of the same general population. A necessary condition for this is that the means and standard deviations for the two groups do not differ beyond the limits prescribed by random sampling.

DISCUSSION

Concerning the Disciplinary Value of Mathematics. Professor R. L. Schuyler, of the department of history in Columbia University, has called my attention to Richard Price's "Observations on the Importance of the American Revolution" [London, 1785, pp. 57-8] where there is found the following notable passage:

"I have in these observations expressed a dislike of systems; but I have meant only to condemn that attachment to them as standards of truth which has been too prevalent. It may be necessary in education to make use of them; or of books explaining them. But they should be used only as guides and helps to enquiry. Instruction in them should be attended with a fair exhibition of the evidence on both sides of every question; and care should be taken to induce, as far as possible, a habit of believing only on an overbalance of evidence; and of proportioning assent in every case to the degree of that overbalance, without regarding authority, antiquity, singularity, novelty, or any of the prejudices which too commonly influence assent. Nothing is so well fitted to produce this habit as the study of *mathematics.* In these sciences no one ever thinks of giving his assent to a proposition till he can clearly understand it, and see it proved by a fair deduction from propositions previously understood and proved. In these sciences the mind is inured to close and patient attention; shown the nature of just reasoning; and taught to form distinct ideas, and to expect clear evidence in all cases before belief. They furnish, therefore, the best exercise for the intellectual powers, and the best defence against that credulity and precipitation and confusion of ideas which are the common sources of error.

There is, however, a danger even here to be avoided. Mathematical studies may absorb the attention too much; and when they do, they contract the mind by rendering it incapable of thinking *at large;* by disqualifying it for judging of any evidence except mathematical; and, consequently, disposing it to an unreasonable scepticism on all subjects which admit not of such evidence. There have been many instances of this narrowness in mathematicians."

CASSIUS J. KEYSER.

NEWS AND NOTES

There is a very active organization of teachers of mathematics in Detroit and its vicinity. Teachers from elementary, intermediate, high schools and junior colleges are represented in the membership of the club. Four times a year the club holds meetings at which well known speakers discuss problems of vital interest. A social half hour precedes each meeting. The program for the current year consists of:

November 23, 1922—Professor H. E. Slaught, University of Chicago: *Mathematics and its Relation to Life*

January 11, 1923—Count Korzybski (author, engineer and social philosopher: *Fate and Freedom*

March 8, 1923—Professor W. B. Ford, University of Michigan, on a subject to be announced later.

April 19, 1923—Professor E. L. Miller, Superintendent of Detroit High Schools: *The Secondary School Curriculum.*

The officers for the year, 1922-1923, are:

President: Henry A. Lane; Vice-Pres.: Erta Curtis; Sec.-Treas.: Selah Warren Mullen.

Professor H. E. Slaught of the University of Chicago spoke before the Detroit Mathematics Club on *Mathematics and its Relation to Life.* Dr. Slaught made the following points:

1. The current layman's notion of mathematics restricts it to the science of quantity, number and space,—or classifies it into arithmetic, algebra, trigonometry, geometry and the calculus.

2. In its higher reaches, mathematics comprises many domains of thought that have nothing to do with quantity, such as the theory of groups; in its higher reaches it includes everything that is logically inferable.

3. Mathematics has for its domain the entire universe with the exception of love, human and divine; an example of mathematics in its higher reaches is the modern theory of cardinal numbers, from the point of view of the logic of classes,—numbers being defined as the class of all classes that have a certain property, defined by a one-to-one relationship.

4. Mathematics requires as much imagination for its successful prosecution, as music, poetry, architecture. If one

should say that architecture is frozen music, the mathematician can reply that geometry is frozen logic. Music is fundamentally mathematical.

5. Mathematics not only proves things; it also proves when things are impossible of solution. In spite of the fact that the tri-section of a general angle is proved impossible by ruler and compass, many ignorant of mathematics spend a lifetime in a vain attempt to achieve its solution. The attempt to square the circle and duplicate the cube are in the same category.

6. Without mathematics, not only would such tangible things as bridges, railroads, and houses disappear, but the whole structure of thought, logic, would be annihilated.

Thus mathematics not only has a relation to life. It is life.

SELAH WARREN MULLEN.

MEETING OF THE MATHEMATICS SECTION CENTRAL ASSOCIATION

Mr. W. G. Gingery, Shortridge High School, Indianapolis, presided at both meetings of the Mathematics Section of the Central Association held in the Hyde Park High School, Chicago, Friday afternoon, December first and Saturday morning, December second.

The first speaker of Friday afternoon was Professor W. W. Hart of the School of Education, University of Wisconsin, whose subject was "The Organization of Secondary Mathematics." His first question was—shall we teach mathematics for mastery or for appreciation? That term "appreciation," so frequently used now-a-days, implies to Mr. Hart merely an "exposure to" the subject, and leads the way to superficiality. Real appreciation can come only through mastery.

To attain that mastery Mr. Hart considers there are five essential points:

1. A minimum list of topics for each grade with supplementary material to suit local needs.
2. Unity, emphasis, and coherence in this material.
3. In each grade a review, preferably indirect, of the essentials of the preceding grades.
4. In each grade a maximum of new material consistent with mastery.

5. Consideration of technique promoting mastery.

Mr. Hart's paper brought forth a most interesting discussion led by Dr. Kinney of Crane Junior College, who advocates the introduction of even the calculus into the secondary course. He thinks that high school pupils are quite capable of doing such work. The climax of the discussion was the motion that a "steering" committee be appointed by the chairman to guide the efforts of the section in this and other important problems.

On Saturday morning Mr. Gingery appointed as members of this "steering" committee with the understanding that they choose their own chairman Mr. Hart, Mr. Reeve of the University of Minnesota, and Mr. Newell of Evanston.

Professor E. J. Moulton, Northwestern University, reported in his paper a most interesting experiment carried on in three departments, mathematics, astronomy, and English. The results of this experiment showed that instructors in mathematics have a greater uniformity in grading papers, both those of other members of the department as well as those which they themselves have previously graded than the teachers of either astronomy or English.

Mr. W. W. Gorsline of Crane Junior College, because of lack of time, could give his audience only a glimpse of the wonderful possibilities of the slide rule in business and in the mathematics class. Slide rules were used by his hearers so that they could more easily follow the solution of the problems.

On Saturday morning, Professor J. R. Overman, State Normal College, Bowling Green, Ohio, read a paper on the training of teachers of mathematics in junior high schools. Professor Overman gave in detail the work both professional and content in either a three-year or a four-year course. The discussion which followed resulted in the section's going on record as favoring the calculus as the minimum in content training for both junior and senior high school teachers.

The following officers were elected: Chairman, Earl G. Thompson, Joliet Township High School; Vice-chairman, A. M. Allison, Lake View High School, Chicago; Secretary, Gertrude L. Anthony, Oak Park and River Forest Township High School.

GERTRUDE L. ANTHONY.

President JOHN H. MINNICK and the Executive Committee of the National Council of Teachers of Mathematics have perfected plans to present the interests of the Council to practically all organizations of mathematics teachers in the United States. State Representatives have been appointed, as follows:

Alabama—Frank Ordway, Central High School, Birmingham
Arizona—A. L. Hartman, Mesa Union High School, Mesa
Arkansas—George W. Drake, University of Arkansas, Fayetteville
California—Gertrude E. Allen, University High School, Oakland
Colorado—E. L. Brown, Northside High School, Denver
Delaware—Mrs. Elinor B. Rosa, Milford
District of Columbia—Harry English, Board of Examiners, Washington
Florida—Miss Olga Larson, Box 84, Apopka
Georgia—George W. Brindle, Surrency
Idaho—Winona M. Perry, 719 Sherman Ave., Couer D'Alene
Illinois—R. L. Modesitt, 1703 S. 7th St., Charleston
Indiana—Walter G. Gingery, Shortridge High School, Indianapolis
Iowa—Ira S. Condit, Iowa State Teachers College, Cedar Falls
Kansas—Miss Inez Morris, 728 State St., Emporia
Kentucky—V. D. Roberts, Somerset
Louisiana—Jeanne Vautrain, 1820 N. Rampan St., New Orleans
Maine—E. L. Moulton, Edward Little High School, Auburn
Maryland—Miss N. V. Orcutt, Girls' Latin High School, Baltimore
Massachusetts—William H. Brown, High School, Amherst
Michigan—John P. Everett, Western State Normal School, Kalamazoo
Minnesota—W. D. Reeve, 828 University Ave., Minneapolis
Mississippi—Miss Clyde Lindsey, Oxford
Missouri—Charles Ammerman, McKinley High School, St. Louis
Nevada—Miss Bertha C. Knemyer, Elko Co., High School, Elko
New Mexico—T. C. Rogers, 1018 Fourth St., E. Las Vegas
New York—Raleigh Schorling, 423 West 123rd St., New York City
Ohio—Miss Florence M. Brooks, Fairmount, Jr. High School, Cleveland
Oklahoma—C. E. Herring, Box 489, Oklahoma City
Oregon—Florence P. Young, Frankline H. S., Portland
Rhode Island—P. S. Crosby, 110 N. Bend St., Pawtucket
South Carolina—J. Bruce Coleman, University of South Carolina, Columbia
South Dakota—Iona J. Rehn, 735 S. Summit Ave., Sioux Falls
Tennessee—F. L. Wrenn, McCallie School, Chattanooga
Texas—J. O. Mahoney, 1900 Crockett St., Dallas
Vermont—Llewellyn R. Perkins, 6 Franklin St., Middlebury
West Virginia—Miss Blanche Stonestreet, 591 Spruce St., Morgantown
Wisconsin—Miss Mary A. Potter, Racine High School, Racine

These representatives are actively engaged in urging the teachers in their respective states to affiliate with the Council, and to participate, in a more direct way, in the reorganization movement now being effected in mathematical education. A special circular has been prepared to set forth the purposes and values of the Council. Copies may be secured from your representative, from Mr. John A. Foberg, Secretary-Treasurer, Camp Hill, Pa., or from President John H. Minnick, School of Education, University of Pennsylvania, Philadelphia.